Problem 2) Substituting $f(x)=x^s\sum_{m=0}^{\infty}a_m x^m$ into the Laguerre equation, we find

$$x\sum_{m=0}^{\infty}a_m(s+m)(s+m-1)x^{s+m-2}+(1+\alpha-x)\sum_{m=0}^{\infty}a_m(s+m)x^{s+m-1}+n\sum_{m=0}^{\infty}a_m x^{s+m}=0.$$

Upon separating the $m=0$ term from two of the infinite sums, the above equation becomes

$$a_0 s(s-1)x^{s-1}+\sum_{m=1}^{\infty}a_m(s+m)(s+m-1)x^{s+m-1}+(1+\alpha)a_0 s x^{s-1}$$
$$+(1+\alpha)\sum_{m=1}^{\infty}a_m(s+m)x^{s+m-1}-\sum_{m=0}^{\infty}a_m(s+m)x^{s+m}+n\sum_{m=0}^{\infty}a_m x^{s+m}=0.$$

The indicial equation, which is readily seen to be $a_0 s(s-1)x^{s-1}+(1+\alpha)a_0 s x^{s-1}=0$, simplifies to

$$a_0 s(s+\alpha)=0.$$

The first two infinite sums can be reorganized by introducing the new dummy $m'=m-1$, then reverting m' back to m. The four summations can then be combined to yield

$$\sum_{m=0}^{\infty}a_{m+1}(s+m+1)(s+m)x^{s+m}+(1+\alpha)\sum_{m=0}^{\infty}a_{m+1}(s+m+1)x^{s+m}$$
$$-\sum_{m=0}^{\infty}a_m(s+m-n)x^{s+m}=0.$$

Setting the coefficient of x^{s+m} to zero (for all m from 0 to ∞), we arrive at the following recursion relation:

$$a_{m+1}(s+m+1)(s+m+1+\alpha)-a_m(s+m-n)=0 \quad\rightarrow\quad a_{m+1}=\frac{s+m-n}{(s+m+1)(s+m+1+\alpha)}a_m.$$

The simplest solution of the indicial equation is $s=0$, which allows a_0 to be chosen arbitrarily while the remaining coefficients are obtained from the recursion relation, as follows:

$$a_{m+1}=\frac{m-n}{(m+1)(m+1+\alpha)}a_m.$$

This recursion relation indicates that the last nonzero coefficient will be a_n, since at $m=n$ the numerator vanishes, at which point a_{n+1} along with all subsequent coefficients a_m become zero. The first few coefficients now reveal the pattern for a general expression of a_m, namely,

$$a_1=-\left(\frac{n}{1+\alpha}\right)a_0 \quad\rightarrow\quad a_2=\frac{n(n-1)}{2(1+\alpha)(2+\alpha)}a_0 \quad\rightarrow\quad a_3=-\frac{n(n-1)(n-2)}{2\cdot 3(1+\alpha)(2+\alpha)(3+\alpha)}a_0 \quad\rightarrow\quad\cdots$$

$$\rightarrow\quad a_m=(-1)^m\frac{n!\,\alpha!}{m!\,(n-m)!(m+\alpha)!}a_0=(-1)^m\frac{(n+\alpha)!}{m!\,(n-m)!(m+\alpha)!}\,\frac{n!\,\alpha!}{(n+\alpha)!}a_0.$$

← For non-integer α, factorial is interpreted in terms of Euler's Gamma function.

Considering that n and α are constant parameters, the polynomial solution for $f(x)$ may be normalized by setting $a_0=\binom{n+\alpha}{n}$, in which case the Laguerre polynomial of order n and parameter α, denoted by $L_n^{\alpha}(x)$, is seen to be

$$L_n^{\alpha}(x)=\sum_{m=0}^{n}(-1)^m\binom{n+\alpha}{n-m}\frac{x^m}{m!}.$$

Digression: The above solution runs into trouble when α is a negative integer, say, $\alpha=-\nu$. In this case, if $1\le\nu\le n$, the recursion relation requires that $a_{\nu-1}$ vanish and, applying the recursion in the reverse direction, one finds that $a_0,a_1,\cdots,a_{\nu-2}$ must vanish as well. The only nonzero coefficients will then be $a_\nu, a_{\nu+1}, \cdots, a_n$. If $\nu \geq n+1$, the above solution for $L_n^\alpha(x)$ retains its validity and, in addition, a new series starts at $m = \nu$ with arbitrary a_ν and continues all the way to infinity, yielding a second solution in the form of an infinite-order polynomial.

As for the second solution $s = -\alpha$ of the indicial equation, several possibilities exist:

i) If α happens to be a non-integer, then all the coefficients a_m obtained from the recursion relation will be nonzero and, of course, proportional to a_0; that is,

$$a_{m+1} = \frac{m-n-\alpha}{(m+1)(m+1-\alpha)} a_m.$$

The second solution of the Laguerre equation for $f(x)$ will then be $x^{-\alpha}$ times a polynomial of infinite-order.

ii) Case of $\alpha = 0$. Here, the second solution of the indicial equation is also $s = 0$, and

$$a_{m+1} = \frac{m-n}{(m+1)^2} a_m.$$

The coefficients a_0 through a_n will be nonzero, and the resulting solution is the same as that obtained in the previous case of $s = 0$.

iii) Case of $\alpha = \nu$, where $\nu > 0$ is an integer. Here, $s = -\nu$ and

$$a_{m+1} = \frac{m-n-\nu}{(m+1)(m+1-\nu)} a_m.$$

In this case, when $m = \nu - 1$, the a_m coefficient must be zero, and using the recursion relation in reverse, one finds that all the coefficients from $a_{\nu-1}$ to a_0 must vanish. Going forward, however, a_ν is arbitrary and $a_{\nu+1}$ to $a_{\nu+n}$ will be found from the recursion relation. The resulting solution ends up being identical to that found in the case of $s = 0$.

iv) Case of $\alpha = -\nu$, where ν is a positive integer. Here, $s = \nu$ and the recursion relation yields

$$a_{m+1} = \frac{m-n+\nu}{(m+1)(m+1+\nu)} a_m.$$

If $1 \leq \nu \leq n$, the coefficients after $a_{n-\nu}$ will vanish, leaving $a_0, a_1, \cdots, a_{n-\nu}$ as the only nonzero coefficients. In contrast, if $\nu \geq n+1$, all the a_m will be nonzero, yielding an infinite-order polynomial solution for the Laguerre equation. Both of these solutions are identical to those obtained previously when $s = 0$ was picked as a solution of the indicial equation.