

**Problem 2)** a, b) Both Fourier integrals can be evaluated by first completing the square in the exponent of the integrand, then switching the real variable  $x$  to the complex variable  $z$  — with the aid of the contours shown in the figure below. We will have

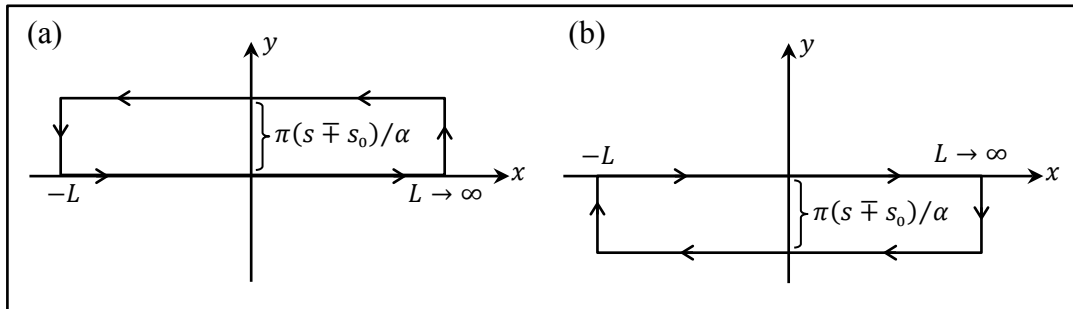
$$\begin{aligned}
 \mathcal{F}\{\exp(-\alpha x^2) \exp(\pm i2\pi s_0 x)\} &= \int_{-\infty}^{\infty} \exp(-\alpha x^2) \exp(\pm i2\pi s_0 x) \exp(-i2\pi s x) dx \\
 &= \int_{-\infty}^{\infty} \exp[-\alpha x^2 - i2\pi(s \mp s_0)x] dx \\
 \text{Completing the square} \rightarrow &= \int_{-\infty}^{\infty} \exp\{-\alpha[x + i\pi\alpha^{-1}(s \mp s_0)]^2 - \pi^2\alpha^{-1}(s \mp s_0)^2\} dx \\
 \text{Use contour in Fig.(a) when } (s \mp s_0) > 0; & \text{ use contour in Fig.(b) when } (s \mp s_0) < 0. \text{ Integrand} \\
 \text{has no poles inside the contour. Vertical legs} & \text{ do not contribute in the limit when } L \rightarrow \infty. \\
 \rightarrow &= \exp\{-[\pi(s \mp s_0)/\sqrt{\alpha}]^2\} \int_{-\infty}^{\infty} \underbrace{\exp\{-\alpha[x + i\pi\alpha^{-1}(s \mp s_0)]^2\}}_z \underbrace{dx}_{dz} \\
 &= \exp\{-[\pi(s \mp s_0)/\sqrt{\alpha}]^2\} \int_{-\infty}^{\infty} \exp(-\alpha x^2) dx \\
 &= \sqrt{\pi/\alpha} \exp\{-\pi[\sqrt{\pi/\alpha}(s \mp s_0)]^2\}. \quad \boxed{= \sqrt{\pi/\alpha}; \text{ see below}} \tag{1}
 \end{aligned}$$

The integral  $\int_{-\infty}^{\infty} \exp(-\alpha x^2) dx$  appearing in the penultimate line of Eq.(1) is evaluated as follows:

$$\begin{aligned}
 \left[\int_{-\infty}^{\infty} \exp(-\alpha x^2) dx\right]^2 &= \iint_{-\infty}^{\infty} \exp[-\alpha(x^2 + y^2)] dx dy \\
 &= \int_0^{\infty} \exp(-\alpha r^2) 2\pi r dr = (\pi/\alpha) \int_0^{\infty} 2x \exp(-x^2) dx = \pi/\alpha. \tag{2}
 \end{aligned}$$

Alternatively, the change of variable  $x = \sqrt{\pi/\alpha} y$  yields

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2) dx = \sqrt{\pi/\alpha} \int_{-\infty}^{\infty} \exp(-\pi y^2) dy = \sqrt{\pi/\alpha}. \tag{3}$$



c) Returning to the final expression in Eq.(1), the Gaussian function  $\exp(-\pi s^2)$  is even and has area equal to 1. Scaling the variable  $s$  by the constant coefficient  $\sqrt{\pi/\alpha}$  narrows the function and reduces its area to  $\sqrt{\alpha/\pi}$ . However, multiplying the function by  $\sqrt{\pi/\alpha}$  restores the area under the function to 1. The argument of the function being  $s \mp s_0$  indicates that the center of the Gaussian has shifted to  $s = \pm s_0$ . We thus have two tall, narrow, and symmetric functions, each having an area equal to 1, one centered at  $s = s_0$ , the other at  $s = -s_0$ . In the limit when  $\alpha \rightarrow 0$ , these become  $\delta(s - s_0)$  and  $\delta(s + s_0)$ , respectively. Considering that  $f(x) = \cos(2\pi s_0 x) = \frac{1}{2}[\exp(i2\pi s_0 x) + \exp(-i2\pi s_0 x)]$ , we conclude that  $F(s) = \frac{1}{2}[\delta(s - s_0) + \delta(s + s_0)]$ .