Problem 2) To substitute $J_n(\alpha x)$ for $f(x)$ in $x^2 f''(x) + x f'(x) + (\alpha^2 x^2 - n^2) f(x)$ we write

$$J_n(\alpha x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x/2)^{2m+n}}{m!(n+m)!},$$

$$J_n'(\alpha x) = (\alpha/2) \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(\alpha x/2)^{2m+n-1}}{m!(n+m)!},$$

$$J_n''(\alpha x) = (\alpha/2)^2 \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(2m+n-1)(\alpha x/2)^{2m+n-2}}{m!(n+m)!}.$$

We will have

$$x^2 J_n''(\alpha x) + x J_n'(\alpha x) + (\alpha^2 x^2 - n^2) J_n(\alpha x)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(2m+n-1)(\alpha x/2)^{2m+n}}{m!(n+m)!} + \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(\alpha x/2)^{2m+n}}{m!(n+m)!}$$

$$+ (\alpha^2 x^2 - n^2) \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x/2)^{2m+n}}{m!(n+m)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m [(2m+n)^2 - n^2](\alpha x/2)^{2m+n}}{m!(n+m)!} + \alpha^2 x^2 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x/2)^{2m+n}}{m!(n+m)!}$$

$$= 4 \sum_{m=0}^{\infty} \frac{(-1)^m m(n+m)(\alpha x/2)^{2m+n}}{m!(n+m)!} + 4 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x/2)^{2m+n+2}}{m!(n+m)!}$$

$$= 4 \sum_{m=1}^{\infty} \frac{(-1)^m (\alpha x/2)^{2m+n}}{(m-1)!(n+m-1)!} + 4 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x/2)^{2m+n+2}}{m!(n+m)!}$$

$k = m-1$ →
$$= 4 \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (\alpha x/2)^{2(k+1)+n}}{k!(n+k)!} + 4 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x/2)^{2m+n+2}}{m!(n+m)!}$$

$$= -4 \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha x/2)^{2k+n+2}}{k!(n+k)!} + 4 \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha x/2)^{2m+n+2}}{m!(n+m)!} = 0.$$

We conclude that $J_n(\alpha x)$ satisfies the Bessel equation $x^2 f''(x) + x f'(x) + (\alpha^2 x^2 - n^2) f(x) = 0$.