

Problem 1) In the method of integration by parts the functions $f(x)$ and $g(x)$ satisfy the identity $\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$.

a) In the present problem, let $f(x) = \ln x$ and $g'(x) = x^n$. We will have

$$\begin{aligned}\int_0^1 x^n \ln(x) dx &= \frac{x^{n+1} \ln x}{n+1} \Big|_0^1 - \frac{1}{n+1} \int_0^1 x^{n+1} x^{-1} dx \\ &= -\frac{1}{n+1} \int_0^1 x^n dx = -\frac{1}{(n+1)^2} x^{n+1} \Big|_0^1 = -\frac{1}{(n+1)^2}.\end{aligned}$$

b) Here $f(x) = \ln x$ and $g'(x) = 1/(1+x)^2$. We will have

$$\begin{aligned}\int_0^1 \frac{\ln x}{(1+x)^2} dx &= \lim_{\varepsilon \rightarrow 0} \left[-\frac{\ln x}{1+x} \Big|_{\varepsilon}^1 + \int_{\varepsilon}^1 \frac{dx}{x(x+1)} \right] = \lim_{\varepsilon \rightarrow 0} \left[\frac{\ln \varepsilon}{1+\varepsilon} + \int_{\varepsilon}^1 \frac{dx}{x} - \int_{\varepsilon}^1 \frac{dx}{x+1} \right] \\ &= \lim_{\varepsilon \rightarrow 0} \left[\frac{\ln \varepsilon}{1+\varepsilon} + \ln x \Big|_{\varepsilon}^1 - \ln(x+1) \Big|_{\varepsilon}^1 \right] \\ &= \lim_{\varepsilon \rightarrow 0} \left[\frac{\ln \varepsilon}{1+\varepsilon} - \ln \varepsilon - \ln 2 + \ln(1+\varepsilon) \right] = -\ln 2.\end{aligned}$$

c) Change of variable: $y = \ln x$. Therefore, $dy = (1/x)dx$, or $dx = \exp(y) dy$. We find

$$\int_0^1 \frac{\ln x}{(1+x)^2} dx = \int_{-\infty}^0 \frac{y \exp(y)}{[1 + \exp(y)]^2} dy = \int_{-\infty}^0 \frac{y}{[\exp(-y/2) + \exp(y/2)]^2} dy$$

$$\boxed{\text{Change of variable: } x = y/2} \rightarrow \int_{-\infty}^0 \frac{x}{\cosh^2(x)} dx \quad \xrightarrow{\boxed{\text{Odd integrand}}} \int_0^{\infty} \frac{x}{\cosh^2(x)} dx = \ln 2.$$