

Problem 1)

a) Using $f(x) = \int_{-\infty}^{\infty} F(s) \exp(i2\pi sx) ds$ and $G(s) = \int_{-\infty}^{\infty} g(x) \exp(-i2\pi sx) dx$, we write

$$\int_{-\infty}^{\infty} f(x) g^*(x) dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} F(s) \exp(i2\pi sx) ds \right) g^*(x) dx$$

Changing order of integration $\rightarrow = \int_{-\infty}^{\infty} F(s) \left(\int_{-\infty}^{\infty} g^*(x) \exp(i2\pi sx) dx \right) ds = \int_{-\infty}^{\infty} F(s) \left(\int_{-\infty}^{\infty} g(x) \exp(-i2\pi sx) dx \right)^* ds$

$$= \int_{-\infty}^{\infty} F(s) G^*(s) ds.$$

b) $\int_{-\infty}^{\infty} \text{sinc}^3(s) ds = \int_{-\infty}^{\infty} \text{sinc}(s) \text{sinc}^2(s) ds = \int_{-\infty}^{\infty} \text{Rect}(x) \text{Tri}(x) dx = 2 \int_0^{1/2} (1-x) dx = \frac{3}{4}.$

c) $\int_{-\infty}^{\infty} \text{sinc}^4(s) ds = \int_{-\infty}^{\infty} \text{sinc}^2(s) \text{sinc}^2(s) ds = \int_{-\infty}^{\infty} \text{Tri}(x) \text{Tri}(x) dx = 2 \int_0^1 (1-x)^2 dx = 2 \int_0^1 y^2 dy = \frac{2}{3}.$

d) $\int_0^{\infty} \exp(-x) \text{sinc}(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} \exp(-|x|) \text{sinc}(x) dx = \int_{-\infty}^{\infty} \frac{\text{Rect}(s)}{1+(2\pi s)^2} ds = 2 \int_0^{1/2} \frac{ds}{1+(2\pi s)^2}$

Change of variable
 $2\pi s = \tan \theta$ $\rightarrow = \frac{2}{2\pi} \int_0^{\tan^{-1}\pi} \frac{(1+\tan^2\theta) d\theta}{1+\tan^2\theta} = \frac{1}{\pi} \int_0^{\tan^{-1}\pi} d\theta = \frac{\tan^{-1}\pi}{\pi}.$