**Problem 5)** $f(x) = (x^2 - 2x + 2)^{-1} = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad (x^2 - 2x + 2)\sum_{n=0}^{\infty} a_n x^n = 1$

$$\rightarrow \quad \left(\sum_{n=0}^{\infty} a_n x^{n+2}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 1$$

$$\rightarrow \quad \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) - 2\left(a_0 x + \sum_{n=2}^{\infty} a_{n-1} x^n\right) + 2\left(a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n\right) = 1$$

$$\rightarrow \quad 2a_0 + 2(a_1 - a_0)x + \sum_{n=2}^{\infty}(a_{n-2} - 2a_{n-1} + 2a_n)x^n = 1$$

$$\rightarrow \quad a_0 = ½, \quad a_1 = a_0 = ½, \quad \boxed{a_n = a_{n-1} - ½a_{n-2} \leftarrow \text{Recursion relation}}$$

$$\rightarrow \quad a_2 = a_1 - ½a_0 = ¼, \quad a_3 = a_2 - ½a_1 = 0, \quad a_4 = -⅛,$$

$$\rightarrow \quad a_5 = -⅛, \quad a_6 = -\tfrac{1}{16}, \quad a_7 = 0, \quad a_8 = \tfrac{1}{32}, \quad a_9 = \tfrac{1}{32},$$

$$\rightarrow \quad a_{10} = \tfrac{1}{64}, \quad a_{11} = 0, \quad a_{12} = -\tfrac{1}{128}, \quad a_{13} = -\tfrac{1}{128}, \quad a_{14} = -\tfrac{1}{256}, \quad \cdots$$

Consequently,

$$(x^2 - 2x + 2)^{-1} = \tfrac{1}{2} + \tfrac{1}{2}x + \tfrac{1}{4}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{8}x^5 - \tfrac{1}{16}x^6 + \tfrac{1}{32}x^8 + \tfrac{1}{32}x^9 + \tfrac{1}{64}x^{10} - \tfrac{1}{128}x^{12} - \tfrac{1}{128}x^{13} - \tfrac{1}{256}x^{14} + \cdots.$$

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**Digression**: Rearranging the above Taylor series, we find

$$(x^2 - 2x + 2)^{-1} = \tfrac{1}{4}(2 + 2x + x^2) - \tfrac{1}{16}(2 + 2x + x^2)x^4 + \tfrac{1}{64}(2 + 2x + x^2)x^8 - \tfrac{1}{256}(2 + 2x + x^2)x^{12} + \cdots$$

$$= \tfrac{1}{4}(x^2 + 2x + 2)\left(1 - \frac{x^4}{4} + \frac{x^8}{16} - \frac{x^{12}}{64} + \cdots\right).$$

The infinite series on the right-hand side of the preceding equation is a geometric series that converges to $1/[1 + (x^4/4)]$, provided that $x^4 < 4$ or $|x| < \sqrt{2}$. The end result is

$$(x^2 - 2x + 2)^{-1} = (x^2 + 2x + 2)/(x^4 + 4),$$

which is readily verified since $(x^2 - 2x + 2)(x^2 + 2x + 2) = (x^2 + 2)^2 - 4x^2 = x^4 + 4$. Note that the radius of convergence of our Taylor series is $\sqrt{2}$. This is because the roots of the polynomial equation $x^2 - 2x + 2 = 0$ are $x_{1,2} = 1 \pm \mathrm{i} = \sqrt{2}\exp(\pm i\pi/4)$. These roots are the poles of the (otherwise analytic) function $f(z) = (z^2 - 2z + 2)^{-1}$ in the complex $z$-plane. The radius of convergence of the corresponding Taylor series in the $z$-plane is the distance from the origin, $z_0 = 0$, to the nearest pole, which, in the present case, is $\sqrt{2}$.

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