**Problem 4**) a) The ellipse equation, $(x/a)^2 + (y/b)^2 = u$, shows that $u$ is non-negative. At the origin, where $x = y = 0$, we have $u = 0$; everywhere else, however, $u$ must be positive.

The hyperbola equation, $(x/c)^2 - (y/d)^2 = v$, shows that at the origin, where $x = y = 0$, we must have $v = 0$. However, $v$ is also zero on the straight lines $y = \pm(d/c)x$.

When $v > 0$, we have $|y| < (d/c)|x|$, so that the contours of constant $v$ lie within the region between the two straight lines $y = \pm(d/c)x$ that contains the $x$-axis.

In contrast, when $v < 0$, we have $|y| > (d/c)|x|$, so that the contours of constant $v$ lie within the region between the two straight lines $y = \pm(d/c)x$ that contains the $y$-axis.

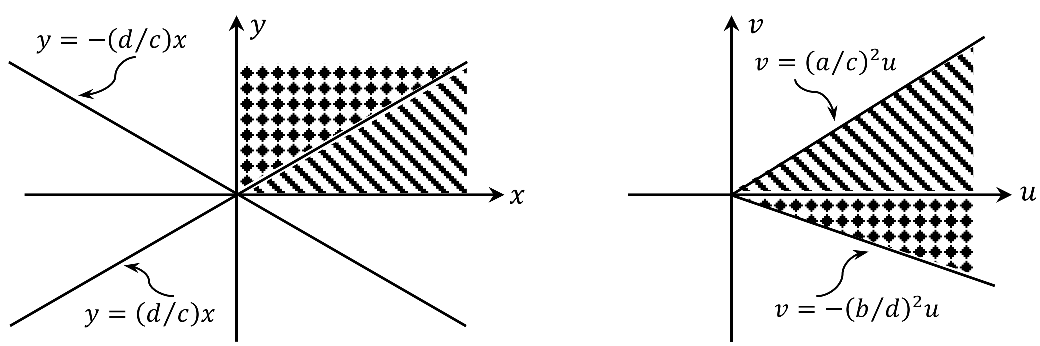
The figure below shows how the first quadrant of the $xy$-plane maps onto the right half of the $uv$-plane. The three remaining quadrants of the $xy$-plane similarly map onto the same region of the $uv$-plane. Note that the left half of the $uv$-plane is not needed at all.

To find the inverse mapping from the $uv$-plane back onto the $xy$-plane, we solve the ellipse and hyperbola equations simultaneously for $x$ and $y$ as functions of $u$ and $v$, as follows:

$$\begin{cases}(b/a)^2x^2 + y^2 = b^2u \\ (d/c)^2x^2 - y^2 = d^2v\end{cases} \quad\rightarrow\quad x^2 = \frac{b^2u + d^2v}{(b/a)^2 + (d/c)^2} = \frac{(ac)^2(b^2u + d^2v)}{(ad)^2 + (bc)^2}. \tag{1}$$

$$\begin{cases}x^2 + (a/b)^2y^2 = a^2u \\ x^2 - (c/d)^2y^2 = c^2v\end{cases} \quad\rightarrow\quad y^2 = \frac{a^2u - c^2v}{(a/b)^2 + (c/d)^2} = \frac{(bd)^2(a^2u - c^2v)}{(ad)^2 + (bc)^2}. \tag{2}$$

It is obvious that $x^2$ and $y^2$ must be non-negative for all values of $u \ge 0$ and $v$. This requires that $-(b/d)^2u \le v \le (a/c)^2u$. In the figure below, the similarly shaded regions depict mappings from different zones within the first quadrant of the $xy$-plane onto the corresponding zones of the $uv$-plane.




b) Being interested here only in the first quadrant of the $xy$-plane, upon taking the square root of Eqs.(1) and (2), we pick the positive solutions for both $x(u, v)$ and $y(u, v)$, as follows:

$$x(u, v) = \frac{ac\sqrt{b^2u + d^2v}}{\sqrt{(ad)^2 + (bc)^2}}, \qquad y(u, v) = \frac{bd\sqrt{a^2u - c^2v}}{\sqrt{(ad)^2 + (bc)^2}}. \tag{3}$$

Partial differentiation now yields

$$\frac{\partial x}{\partial u} = \frac{ac}{\sqrt{(ad)^2 + (bc)^2}} \times \frac{b^2}{2\sqrt{b^2u + d^2v}}, \qquad \frac{\partial y}{\partial u} = \frac{bd}{\sqrt{(ad)^2 + (bc)^2}} \times \frac{a^2}{2\sqrt{a^2u - c^2v}}, \tag{4}$$

$$\frac{\partial x}{\partial v} = \frac{ac}{\sqrt{(ad)^2 + (bc)^2}} \times \frac{d^2}{2\sqrt{b^2u + d^2v}}; \qquad \frac{\partial y}{\partial v} = -\frac{bd}{\sqrt{(ad)^2 + (bc)^2}} \times \frac{c^2}{2\sqrt{a^2u - c^2v}}. \tag{5}$$

Subsequently, the Jacobian is found to be

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{Vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{Vmatrix} = \frac{abcd}{4\sqrt{(a^2u - c^2v)(b^2u + d^2v)}}. \tag{6}$$

**Digression**: An alternative method of computing the Jacobian is to evaluate $\partial(u,v)/\partial(x,y)$ at first, then invert the result, as follows:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{Vmatrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{Vmatrix} = \begin{Vmatrix} 2x/a^2 & 2x/c^2 \\ 2y/b^2 & -2y/d^2 \end{Vmatrix} = \frac{4xy}{(ad)^2} + \frac{4xy}{(bc)^2} = \frac{4xy[(bc)^2 + (ad)^2]}{(abcd)^2}$$

$$= \frac{4abcd\sqrt{(a^2u - c^2v)(b^2u + d^2v)}}{(ad)^2 + (bc)^2} \times \frac{(bc)^2 + (ad)^2}{(abcd)^2} = \frac{4\sqrt{(a^2u - c^2v)(b^2u + d^2v)}}{abcd}. \tag{7}$$

Clearly, the Jacobian in Eq.(6) for transforming from $xy$ to the $uv$ coordinates is the inverse of that in Eq.(7) for transforming from the $uv$ back into the $xy$ coordinate system.