Solution to Problem 4)

a) $\int_1^{N+1} x^{k-1}\mathrm{d}x = \sum_{n=1}^{N}\int_n^{n+1} x^{k-1}\mathrm{d}x \quad\rightarrow\quad \frac{1}{k}[(N+1)^k - 1] = \frac{1}{k}\sum_{n=1}^{N}[(n+1)^k - n^k]$

$$\rightarrow\ (N+1)^k - 1 = \sum_{n=1}^{N}\left[\sum_{m=0}^{k}\binom{k}{m}n^m - n^k\right] = \sum_{n=1}^{N}\sum_{m=0}^{k-1}\binom{k}{m}n^m = \sum_{m=0}^{k-1}\binom{k}{m}\sum_{n=1}^{N}n^m$$

$$\rightarrow\ \sum_{m=0}^{k-1}\binom{k}{m}S_N^{(m)} = (N+1)^k - 1.$$

b) $k=1$: $S_N^{(0)} = N$ (that is, $1^0 + 2^0 + 3^0 + \cdots + N^0 = N$).

$k=2$: $S_N^{(0)} + \binom{2}{1}S_N^{(1)} = (N+1)^2 - 1 \quad\rightarrow\quad S_N^{(1)} = N(N+1)/2.$

$k=3$: $S_N^{(0)} + \binom{3}{1}S_N^{(1)} + \binom{3}{2}S_N^{(2)} = (N+1)^3 - 1$

$\rightarrow\ N + 3N(N+1)/2 + 3S_N^{(2)} = N^3 + 3N^2 + 3N$

$\rightarrow\ S_N^{(2)} = N(2N^2 + 3N + 1)/6 = N(N+1)(2N+1)/6.$

$k=4$: $S_N^{(0)} + \binom{4}{1}S_N^{(1)} + \binom{4}{2}S_N^{(2)} + \binom{4}{3}S_N^{(3)} = (N+1)^4 - 1$

$\rightarrow\ N + 2N(N+1) + N(N+1)(2N+1) + 4S_N^{(3)} = N^4 + 4N^3 + 6N^2 + 4N$

$\rightarrow\ S_N^{(3)} = [N(N+1)/2]^2.$

Repeating the procedure for successive values of k enables one to obtain closed form expressions for $S_N^{(m)}$ corresponding to larger values of m.