**Solution to Problem 4)**

a)
$$\ln(x+1)|_{x=0} = \ln 1 = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x+1) = \frac{1}{x+1}\Big|_{x=0} = 1,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln(x+1) = \frac{\mathrm{d}}{\mathrm{d}x}(x+1)^{-1} = -(x+1)^{-2}|_{x=0} = -1,$$
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\ln(x+1) = -\frac{\mathrm{d}}{\mathrm{d}x}(x+1)^{-2} = 2(x+1)^{-3}|_{x=0} = 2,$$
$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}\ln(x+1) = \frac{\mathrm{d}}{\mathrm{d}x}2(x+1)^{-3} = -3!\,(x+1)^{-4}|_{x=0} = -3!,$$
$$\frac{\mathrm{d}^5}{\mathrm{d}x^5}\ln(x+1) = -3!\frac{\mathrm{d}}{\mathrm{d}x}(x+1)^{-4} = 4!\,(x+1)^{-5}|_{x=0} = 4!.$$

Therefore, $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty}[(-1)^{n+1}/n]\,x^n$. Substituting for $\ln(x+1)$ in $f(x)$ now yields

$$f(x) = \frac{x}{\ln(x+1)} = \frac{x}{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots} = \frac{1}{1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \cdots} = 1/\sum_{n=0}^{\infty}[(-1)^n/(n+1)]\,x^n.$$

It is now easy to see that $f(x=0) = 1$ and $f(x=-1) = \frac{1}{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots} = \frac{1}{\infty} = 0$.

b) $f(x) = 1/\sum_{n=0}^{\infty}[(-1)^n/(n+1)]\,x^n = \sum_{m=0}^{\infty} a_m x^m$

$\rightarrow \quad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[(-1)^n a_m/(n+1)]\,x^{n+m} = 1$

$\rightarrow \quad \sum_{k=0}^{\infty}\sum_{n=0}^{k}[(-1)^n a_{k-n}/(n+1)]\,x^k = 1$ ← Defining $k = m + n$

Setting $k = 0$, the above equation yields the value of the first coefficient as $a_0 = 1$. For $k \ge 1$, the coefficient of $x^k$ must be zero, that is, $\sum_{n=0}^{k}[(-1)^n a_{k-n}/(n+1)] = 0$. We thus find

$k = 1$: $\quad a_1 - \frac{1}{2}a_0 = 0 \quad \rightarrow \quad a_1 = \frac{1}{2}$.

$k = 2$: $\quad a_2 - \frac{1}{2}a_1 + \frac{1}{3}a_0 = 0 \quad \rightarrow \quad a_2 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$.

$k = 3$: $\quad a_3 - \frac{1}{2}a_2 + \frac{1}{3}a_1 - \frac{1}{4}a_0 = 0 \quad \rightarrow \quad a_3 = -\frac{1}{24} - \frac{1}{6} + \frac{1}{4} = \frac{1}{24}$.

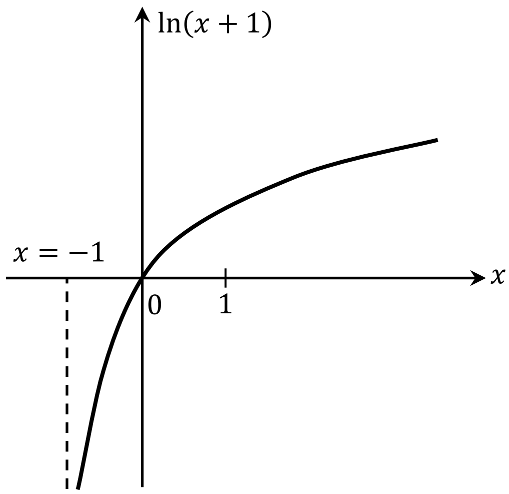
$k = 4$: $\quad a_4 - \frac{1}{2}a_3 + \frac{1}{3}a_2 - \frac{1}{4}a_1 + \frac{1}{5}a_0 = 0 \quad \rightarrow \quad a_4 = \frac{1}{48} + \frac{1}{36} + \frac{1}{8} - \frac{1}{5} = -\frac{19}{720}$.

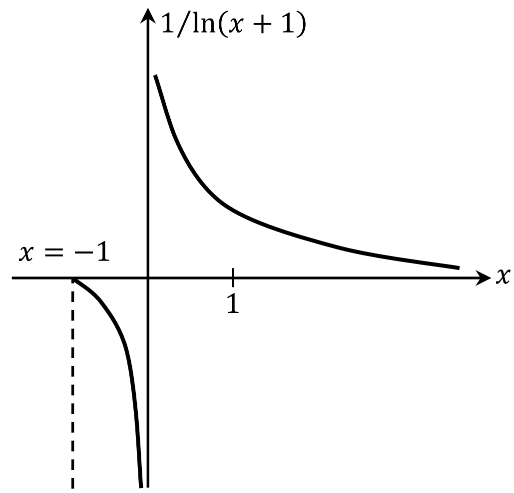
The Taylor series expansion of $f(x)$ is thus given by

$$f(x) = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \cdots.$$

As a check on the validity of this expansion, note that the first five terms of the above Taylor series yield $f(-1) \cong 0.3486$, $f(-\frac{1}{2}) \cong 0.7223$, $f(0) = 1$, $f(\frac{1}{2}) \cong 1.2327$, $f(1) \cong 1.4319$, while the actual values of the function are $f(-1) = 0$, $f(-½) = 0.7213\cdots$, $f(0) = 1$, $f(½) = 1.2331\cdots$, and $f(1) = 1.4426\cdots$.

c) The functions $\ln(x+1)$ and $1/\ln(x+1)$ are plotted below.




The function $f(x) = x/\ln(x+1)$ is well defined over its domain $x \geq -1$, even at the ambiguous points $x = -1$ and $x = 0$; see part (a). The slope of the function at the ambiguous points is $f'(x = -1) = \infty$ and $f'(x = 0) = ½$. A plot of $f(x)$ appears below.