Problem 3) a) The minor and major semi-axes of the cross-sectional ellipse at a height $z = n\Delta$ are $(n\Delta/h)a$ and $(n\Delta/h)b$. Therefore, assuming the slice is sufficiently thin, the volume of the corresponding slice will be $\pi(n\Delta/h)^2 ab\Delta$. The volume of the cone is now obtained by adding up the volumes of all the slices from $n = 1$ to $N = h/\Delta$, as follows:

$$V = lim_{\Delta\to 0} \sum_{n=1}^{N} \pi(n\Delta/h)^2 ab\Delta = lim_{\Delta\to 0}(\pi ab\Delta^3/h^2) \sum_{n=1}^{N} n^2$$

$$= lim_{\Delta\to 0}(\pi ab\Delta^3/6h^2)N(N+1)(2N+1)$$

$$= lim_{\Delta\to 0}(\pi ab/6h^2)h(h+\Delta)(2h+\Delta) = \pi abh/3.$$

b) The procedure is the same, except that the minor and major semi-axes of the cross-sectional ellipse at $z = n\Delta$ are now $(\sqrt{n\Delta/h})a$ and $(\sqrt{n\Delta/h})b$. The total volume is thus given by

$$V = lim_{\Delta\to 0} \sum_{n=1}^{N} \pi(n\Delta/h)ab\Delta = lim_{\Delta\to 0}(\pi ab\Delta^2/h) \sum_{n=1}^{N} n$$

$$= lim_{\Delta\to 0}(\pi ab\Delta^2/2h)N(N+1) = lim_{\Delta\to 0}(\pi ab/2h)h(h+\Delta) = \pi abh/2.$$

c) Once again, the procedure is the same, but the minor and major semi-axes of the cross-sectional ellipse at $z = n\Delta$ are $(n\Delta/h)^{3/2}a$ and $(n\Delta/h)^{3/2}b$. The total volume is now given by

$$V = lim_{\Delta\to 0} \sum_{n=1}^{N} \pi(n\Delta/h)^3 ab\Delta = lim_{\Delta\to 0}(\pi ab\Delta^4/h^3) \sum_{n=1}^{N} n^3$$

$$= lim_{\Delta\to 0}(\pi ab\Delta^4/h^3)[N(N+1)/2]^2 = lim_{\Delta\to 0}(\pi ab/4h^3)[h(h+\Delta)]^2 = \pi abh/4.$$