Problem 3) a) Squaring $f(x)$, then solving the resulting quadratic equation, yields a compact expression for $f(x)$, as follows:

$$f^2(x) = x + f(x) \quad \rightarrow \quad f^2(x) - f(x) - x = 0 \quad \rightarrow \quad f(x) = ½ \pm \sqrt{¼ + x}.$$

b) For $x > 0$, the solution with the minus sign yields a negative value for $f(x)$, which is obviously unacceptable. Therefore,

$$f(x) = ½\big(1 + \sqrt{1 + 4x}\big); \qquad\qquad x > 0.$$

For small values of x, the above equation yields $f(x) \cong ½ + ½(1 + 2x) = 1 + x$, which approaches 1.0 as $x \rightarrow 0$, despite the fact that direct evaluation yields $f(0) = 0$. This indicates a discontinuity of $f(x)$ at $x = 0$. For $x = ½$, we find $f(x) = (1 + \sqrt{3})/2$, and for $x = 1$, we find $f(x) = (1 + \sqrt{5})/2$. For large values of x, we will have $f(x) \cong ½ + \sqrt{x}$.

c) At $x = 0$, direct evaluation yields $f(x) = 0$ and, therefore, of the two solutions found in (a), the correct solution is the one with a minus sign. This is yet another indication that $f(x)$ is discontinuous at $x = 0$.

d) For $x < 0$, each radical must be treated as the square root of a complex number. For the method employed in part (a) to yield a correct answer, it is necessary to *consistently* pick, for all the various radicals, one of the two possible values of the square root. Depending on which root is (consistently) selected, we obtain one or the other solution obtained in part (a). For $x = -¼$, the two solutions coincide at $f(x) = ½$. For all other negative values of x, we obtain two solutions for $f(x)$, both of which are acceptable. For instance, $f(-⅛) = ½(1 \pm \sqrt{2}/2)$, and $f(-½) = (1 \pm \mathrm{i})/2$. Note that, for x in the interval $[-¼, 0)$, both solutions are real, whereas for $x < -¼$, both solutions are complex.

e) For complex-valued x, we obtain the same two solutions as in part (a). Both solutions are acceptable provided that we *consistently* pick one of the two possible values for each square root.