Problem 3) Suppose that $\sqrt{N} = m/n$, where both m and n are integers. Assume further that m and n lack common divisors. Write $m = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$ and $n = q_1^{\ell_1} q_2^{\ell_2} \cdots q_j^{\ell_j}$. If N happens to be a perfect square, then $\sqrt{N}$ will be an integer, in which case $j = 1$, $q_1 = 1$, and there is nothing else to say. Otherwise, we have $m^2 = Nn^2$, which may be written as follows:

$$p_1^{2k_1} p_2^{2k_2} \cdots p_i^{2k_i} = N q_1^{2\ell_1} q_2^{2\ell_2} \cdots q_j^{2\ell_j}.$$

It is clear that the above equality cannot be satisfied, no matter what prime factors happen to reside within N. The prime factors of N cannot eliminate any of the qs appearing on the right-hand side of the above equation, and unless these factors are cancelled out, the equality cannot hold. We thus have a contradiction, which is rooted in the fact that we began by assuming that $\sqrt{N}$ is rational. The conclusion is that, unless N is a perfect square, $\sqrt{N}$ will be irrational.