Solution to Problem 2) If the pyramid is imagined to have been cut with a plane parallel to the xy-plane at a height z, its cross-section at the location of the cut will have the same shape as the base of the pyramid, albeit with an area that is reduced by a factor $(1-z/h)^2$. The volume confined between two planes located at z and $z+\mathrm{d}z$ will thus be $A(1-z/h)^2\mathrm{d}z$. The volume of the pyramid is found by integrating this differential volume from $z=0$ to $z=h$. We will have

$$V=\int_0^h A(1-z/h)^2\mathrm{d}z = Ah\int_0^1(1-\zeta)^2\mathrm{d}\zeta = -⅓Ah(1-\zeta)^3|_{\zeta=0}^1 = ⅓Ah. \quad (1)$$

Note that the shape of the base as well as its location and orientation within the xy-plane are totally irrelevant. Moreover, the base can be an infinite-sided polygon, in which case it could acquire a smooth shape such as a circle, an ellipse, etc. The volume of the pyramid (or cone) is always going to be ⅓ the area A of the base times the height h.

An alternative method of solving this problem is to imagine the pyramid sliced into a large number N of thin layers, each having thickness h/N and cross-sectional area $A(n/N)^2$, with n ranging from 1 to N. The total volume of the pyramid will then be

$$V=\lim_{N\to\infty}\sum_{n=1}^N A(n/N)^2(h/N) = lim_{N\to\infty}(Ah/N^3)\sum_{n=1}^N n^2$$ ← see chapter 1, problem 7

$$= lim_{N\to\infty}(Ah/N^3)\,[N(N+1)(2N+1)/6]$$

$$= lim_{N\to\infty} Ah\,(1+N^{-1})(2+N^{-1})/6 = ⅓Ah. \quad (2)$$

b) For a truncated pyramid, the upper limit of the integral in Eq.(1) will be αh. We will have

$$V=\int_0^{\alpha h} A(1-z/h)^2\mathrm{d}z = Ah\int_0^\alpha(1-\zeta)^2\mathrm{d}\zeta = -⅓Ah(1-\zeta)^3|_{\zeta=0}^\alpha$$

$$= ⅓Ah[1-(1-\alpha)^3] = A\alpha h(1-\alpha+\alpha^2/3). \quad (3)$$

Another way of answering this question is by noting that the volume removed from the top of the pyramid has a base area $(1-\alpha)^2A$ and a height $(1-\alpha)h$. Consequently, the removed volume is $⅓Ah(1-\alpha)^3$. The remaining volume is, therefore, given by $⅓Ah[1-(1-\alpha)^3]$, which is the same as that given by Eq.(3).