Solution to Problem 2) a) For the reflected light, the first reflection coefficient at the interface between media 1 and 2 is ρ_{12}. The ray then enters the layer (which entails multiplication by τ_{12}), propagates to the bottom of the layer (multiplication by η), gets reflected at the $2-3$ interface (multiplication by ρ_{23}), returns to the top of the thin layer (multiplication by η), and gets transmitted to the incidence medium (multiplication by τ_{21}). The process then continues, with each additional round trip through the thin layer involving another multiplication by $\rho_{21}\eta\rho_{23}\eta$. The overall reflection coefficient r at the top surface of the thin layer is thus given by the following infinite series:

$$\begin{aligned} r &= \rho_{12} + \tau_{12}(\eta\rho_{23}\eta)\tau_{21} + \tau_{12}(\eta\rho_{23}\eta)(\rho_{21}\eta\rho_{23}\eta)\tau_{21} + \tau_{12}(\eta\rho_{23}\eta)(\rho_{21}\eta\rho_{23}\eta)^2\tau_{21} + \cdots \\ &= \rho_{12} + \tau_{12}\rho_{23}\eta^2\tau_{21}[1 + (\rho_{21}\rho_{23}\eta^2) + (\rho_{21}\rho_{23}\eta^2)^2 + (\rho_{21}\rho_{23}\eta^2)^3 + \cdots]. \end{aligned}$$

As for the transmitted light, the incident ray must enter the top layer (multiplication by τ_{12}), propagate to the bottom of the layer (multiplication by η), then exit from medium 2 to medium 3 (multiplication by τ_{23}). The ray arriving at the $2-3$ interface also gets reflected (multiplication by ρ_{23}), propagates upward (multiplication by η), bounces back from the $2-1$ interface (multiplication by ρ_{21}), propagates to the bottom of the thin layer (multiplication by η), then exits at the $2-3$ interface (multiplication by τ_{23}). The process continues, with each round trip involving a multiplication by $\rho_{23}\eta\rho_{21}\eta$. The overall transmission coefficient t at the bottom of the thin layer is thus given by the following infinite series:

$$\begin{aligned} t &= \tau_{12}\eta\tau_{23} + \tau_{12}\eta(\rho_{23}\eta\rho_{21}\eta)\tau_{23} + \tau_{12}\eta(\rho_{23}\eta\rho_{21}\eta)^2\tau_{23} + \tau_{12}\eta(\rho_{23}\eta\rho_{21}\eta)^3\tau_{23} + \cdots \\ &= \tau_{12}\eta\tau_{23}[1 + (\rho_{23}\rho_{21}\eta^2) + (\rho_{23}\rho_{21}\eta^2)^2 + (\rho_{23}\rho_{21}\eta^2)^3 + \cdots]. \end{aligned}$$

b) We may now use the geometric series formula to express the total reflection and transmission coefficients in closed form, that is,

$$r = \rho_{12} + \frac{\tau_{12}\rho_{23}\eta^2\tau_{21}}{1-\rho_{21}\rho_{23}\eta^2} = \frac{\rho_{12} - (\rho_{12}\rho_{21} - \tau_{12}\tau_{21})\rho_{23}\eta^2}{1-\rho_{21}\rho_{23}\eta^2}.$$

$$t = \frac{\tau_{12}\tau_{23}\eta}{1-\rho_{21}\rho_{23}\eta^2}.$$