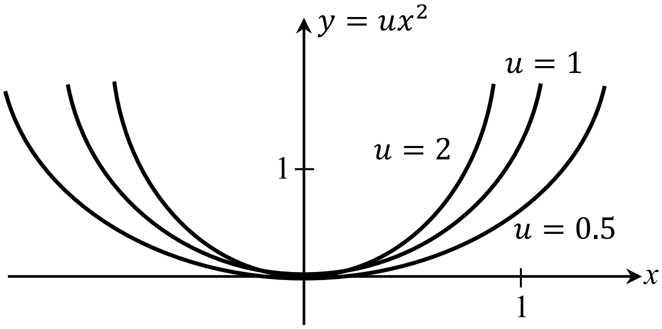
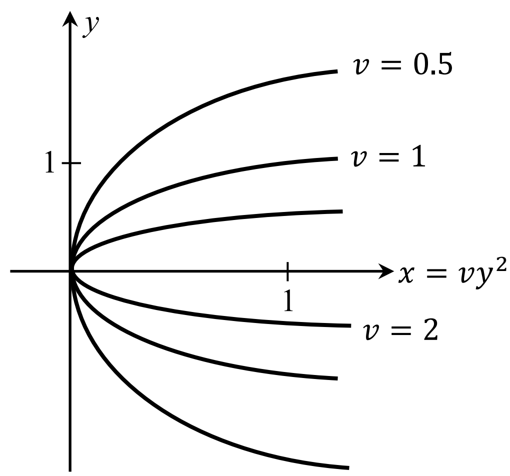
Solution to Problem 2) a) The following figures show several contours of constant u and constant v within the xy-plane. (Similar curves can be drawn for negative values of u and v as well.) Note that, within each quadrant of the xy-plane, the contours of constant u cover the entire quadrant (except for the y-axis), the contours of constant v also cover the entire quadrant (except for the x-axis), and, aside from the point at the origin, there exists only one point in any given quadrant where a constant-u contour crosses a constant-v contour. The crossing point identifies the unique Cartesian (x, y) coordinates associated with the curvilinear (u, v) coordinates.





b) $u = y/x^2 \quad \rightarrow \quad \partial u/\partial x = -2y/x^3 \quad \text{and} \quad \partial u/\partial y = 1/x^2,$
$v = x/y^2 \quad \rightarrow \quad \partial v/\partial x = 1/y^2 \quad \text{and} \quad \partial v/\partial y = -2x/y^3.$

Jacobian: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} -2y/x^3 & 1/y^2 \\ 1/x^2 & -2x/y^3 \end{vmatrix} = \frac{3}{x^2y^2}.$

Similarly,

$x = vy^2 = vu^2x^4 \quad \rightarrow \quad x = u^{-2/3}v^{-1/3},$
$y = ux^2 = uv^2y^4 \quad \rightarrow \quad y = u^{-1/3}v^{-2/3}.$

$\partial x/\partial u = -\tfrac{2}{3}u^{-5/3}v^{-1/3} \quad \text{and} \quad \partial x/\partial v = -\tfrac{1}{3}u^{-2/3}v^{-4/3},$

$\partial y/\partial u = -\tfrac{1}{3}u^{-4/3}v^{-2/3} \quad \text{and} \quad \partial y/\partial v = -\tfrac{2}{3}u^{-1/3}v^{-5/3}.$

Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} -\tfrac{2}{3}u^{-5/3}v^{-1/3} & -\tfrac{1}{3}u^{-4/3}v^{-2/3} \\ -\tfrac{1}{3}u^{-2/3}v^{-4/3} & -\tfrac{2}{3}u^{-1/3}v^{-5/3} \end{vmatrix} = \frac{1}{3u^2v^2}.$

Considering that $uv = 1/(xy)$, it is readily seen that $\partial(u, v)/\partial(x, y)$ and $\partial(x, y)/\partial(u, v)$ are inverses of each other.

c) In the xy-plane, the shaded area is determined as follows:

$$\text{Area} = \int_{x=0}^{1} \left(\int_{y=x^2}^{\sqrt{x}} \mathrm{d}y\right) \mathrm{d}x = \int_0^1 \left(\sqrt{x} - x^2\right)\mathrm{d}x = \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)_{x=0}^{1} = 2/3 - 1/3 = 1/3.$$

The same area may be computed in the uv coordinate system by recognizing that both u and v range from 1.0 to ∞, and that the relevant Jacobian is $|\partial(x, y)/\partial(u, v)| = 1/(3u^2v^2)$.

$$\text{Area} = \int_{u=1}^{\infty} \int_{v=1}^{\infty} 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v = 1/3 \int_1^{\infty} \frac{\mathrm{d}u}{u^2} \int_1^{\infty} \frac{\mathrm{d}v}{v^2} = 1/3 \left(-\frac{1}{u}\right)_{u=1}^{\infty} \left(-\frac{1}{v}\right)_{v=1}^{\infty} = 1/3.$$
