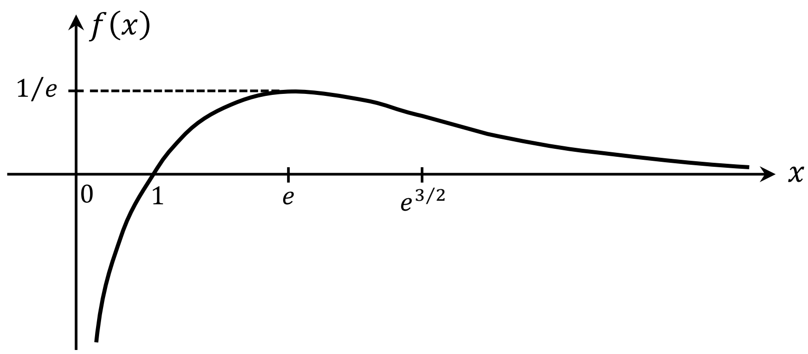
Problem 1) $f(x) = x^{-1}\ln x \quad \rightarrow \quad f'(x) = -x^{-2}\ln x + x^{-1}(1/x) = (1-\ln x)/x^2 = 0.$

The solution to this equation is $\ln x = 1$, or $x = e$. To determine if this is a maximum, minimum, or inflection point of the function, we must evaluate the 2nd derivative of $f(x)$ at $x = e$. We find

$$f''(x) = [-(1/x)x^2 - 2x(1-\ln x)]/x^4 = (2\ln x - 3)/x^3 \quad \rightarrow \quad f''(e) = -1/e^3.$$

Since the 2nd derivative of $f(x)$ at $x = e$ is negative, the function has a maximum at this point. Noting that $x^{-1}\ln x = 0$ at $x = 1$, that the function goes to $-\infty$ when $x \rightarrow 0$, peaks at $x = e$, reverses its curvature at $x = e^{3/2}$, and goes to zero when $x \rightarrow \infty$, we can plot $f(x)$ as follows:



To integrate $f(x) = x^{-1}\ln x$, we note that x^{-1} is the derivative of $\ln x$; therefore,

$$\int_1^{x_0} x^{-1}\ln x\,\mathrm{d}x = ½\ln^2 x\Big|_{x=1}^{x_0} = ½\ln^2(x_0).$$

Alternatively, one may use the method of integration by parts to arrive at the same result; that is,

$$\int_1^{x_0} x^{-1}\ln x\,\mathrm{d}x = (\ln x)(\ln x)\Big|_{x=1}^{x_0} - \int_1^{x_0}(\ln x)(x^{-1})\mathrm{d}x$$

$$\rightarrow \quad \int_1^{x_0} x^{-1}\ln x\,\mathrm{d}x + \int_1^{x_0} x^{-1}\ln x\,\mathrm{d}x = \ln^2(x_0) \quad \rightarrow \quad \int_1^{x_0} x^{-1}\ln x\,\mathrm{d}x = ½\ln^2(x_0).$$

Note that the area under the positive lobe of the function from $x = 1$ to $x_0 \geq 1$ is equal in magnitude and opposite in sign to the area under the negative lobe from $x = x_0^{-1}$ to 1. The areas under both lobes go to infinity when $x_0 \rightarrow 0$ (negative lobe) and when $x_0 \rightarrow \infty$ (positive lobe).