Solution to Problem 1) The time taken by a light ray to travel from the object to the point $(x, 0, 0)$ on the surface of the water, and from there to the observer's eye, is denoted by $t(x)$ and is given by

$$t(x) = (n_1/c)\sqrt{(\ell - x)^2 + h^2} + (n_0/c)\sqrt{x^2 + d^2}. \tag{1}$$

To minimize $t(x)$, we set its derivative with respect to x equal to zero, that is,

$$\frac{\mathrm{d}t(x)}{\mathrm{d}x} = -\frac{(n_1/c)(\ell - x)}{\sqrt{(\ell-x)^2 + h^2}} + \frac{(n_0/c)x}{\sqrt{x^2 + d^2}} = 0. \tag{2}$$

Considering that $x/\sqrt{x^2 + d^2} = \sin\theta_0$ and $(\ell - x)/\sqrt{(\ell - x)^2 + h^2} = \sin\theta_1$, an immediate consequence of Eq.(2) is that

$$n_1 \sin\theta_1 = n_0 \sin\theta_0. \tag{3}$$

The above equation, of course, is the well-known Snell's law of refraction. To confirm that the above solution does, in fact, represent a *minimum* of $t(x)$, we evaluate the second derivative $t''(x)$ of $t(x)$, as follows:

$$\begin{aligned} t''(x) &= (n_1/c)[(\ell - x)^2 + h^2]^{-1/2} - (n_1/c)(\ell - x)^2[(\ell - x)^2 + h^2]^{-3/2} \\ &\quad +(n_0/c)(x^2 + d^2)^{-1/2} - (n_0/c)x^2(x^2 + d^2)^{-3/2} \\ &= \frac{n_1 h^2}{c[(\ell-x)^2 + h^2]^{3/2}} + \frac{n_0 d^2}{c(x^2+d^2)^{3/2}}. \end{aligned} \tag{4}$$

Clearly, $t''(x) > 0$ for all values of x and, therefore, the solution of Eq.(2) given by Eq.(3) represents a minimum of $t(x)$. A less mathematical but perhaps more intuitive approach to this problem would involve plotting the general behavior of the function $t(x)$ of Eq.(1), then arguing that its only minimum must occur somewhere between $x = 0$ and $x = \ell$.

Digression: Equation (2) may be further simplified and written as follows:

$$\begin{aligned} &\frac{n_1^2(\ell - x)^2}{(\ell-x)^2 + h^2} = \frac{n_0^2 x^2}{x^2 + d^2} \\ \rightarrow\quad &1 + \left(\frac{h}{\ell-x}\right)^2 = (n_1/n_0)^2[1 + (d/x)^2] \\ \rightarrow\quad &\left(\frac{h}{\ell-x}\right)^2 - (n_1/n_0)^2(d/x)^2 = (n_1/n_0)^2 - 1 \\ \rightarrow\quad &h^2x^2 - (n_1 d/n_0)^2(\ell - x)^2 = [(n_1/n_0)^2 - 1]x^2(\ell - x)^2 \\ \rightarrow\quad &(x - \ell)^2\{[(n_1/n_0)^2 - 1]x^2 + (n_1 d/n_0)^2\} - h^2x^2 = 0. \end{aligned} \tag{5}$$

This is a quartic (or 4th degree polynomial) equation in x. Explicit formulas exist for solving quartic equations, although, in general, the procedure is long and tedious. Note that the above quartic equation has four solutions, of which only one satisfies Eq.(2). Of the remaining three solutions, at least one must be real-valued (because the coefficients of the quartic are all real, and the equation is already known to have one real-valued solution, which is given by Eq.(3)). The remaining solutions of Eq.(5) are either complex-valued (in the form of a complex-conjugate pair) or fall outside the $[0, \ell]$ interval. In the latter case, the solution will fail to satisfy Eq.(2). Moreover, the solution given by Eq.(3) cannot be a repeated solution, because a repeated solution would make $t''(x)$ equal to zero, which Eq.(4) does not allow.
