Opti 503 **Midterm Exam** (4/11/2011) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

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3 pts **Problem 1**) a) Using the scaling and differentiation theorems, find the Fourier transform of $f(x) = x\,\text{Rect}(x/2)$.

3 pts b) Verify that the result obtained in (a) is in agreement with the function $F(s) = \mathscr{F}\{f(x)\}$ obtained by direct integration.

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3 pts **Problem 2**) a) Plot the function $f(x) = \text{comb}(x)\cos(\pi x)$, find its Fourier transform $F(s)$, then plot the function $F(s)$.

3 pts b) Plot the function $g(x) = \text{comb}(x - \tfrac{1}{2})$, find its Fourier transform $G(s)$, then plot the function $G(s)$.

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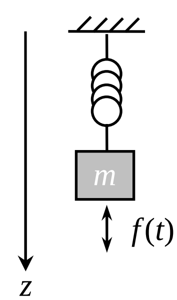
6 pts **Problem 3**) An integral representation of $J_0(x)$, the Bessel function of first kind, 0^{th} order, is

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(\mathrm{i}x\sin\theta)\,\mathrm{d}\theta.$$

Using the method of stationary-phase approximation, show that the asymptotic form of $J_0(x)$ for large values of x is $J_0(x) \sim \sqrt{\dfrac{2}{\pi x}}\cos(x - \tfrac{\pi}{4})$.

Problem 4) A critically-damped mass-and-spring system is driven by a sinusoidal force of amplitude F_o and frequency f_o, that is, $f(t) = F_o\,\text{Step}(t)\sin(2\pi f_o t)$; here both F_o and f_o are real-valued. The second-order differential equation governing the system is thus given by

$$\frac{\mathrm{d}^2 z_1(t)}{\mathrm{d}t^2} + 2\omega_o \frac{\mathrm{d}z_1(t)}{\mathrm{d}t} + \omega_o^2 z_1(t) = (F_o/m)\,\text{Step}(t)\sin(2\pi f_o t).$$



As usual, m is the mass and ω_o the natural frequency of the oscillator. Note that, since the system is assumed to be critically-damped, the damping coefficient γ is set equal to $2\omega_o$.

5 pts a) Use the Fourier transform method to determine the displacement function $z_1(t)$ at all times t.

2 pts b) Describe the behavior of the system as the excitation frequency approaches the natural frequency of the system, that is, as $2\pi f_o \to \omega_o$.
