Opti 503A **Midterm Exam** (3/16/2020) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

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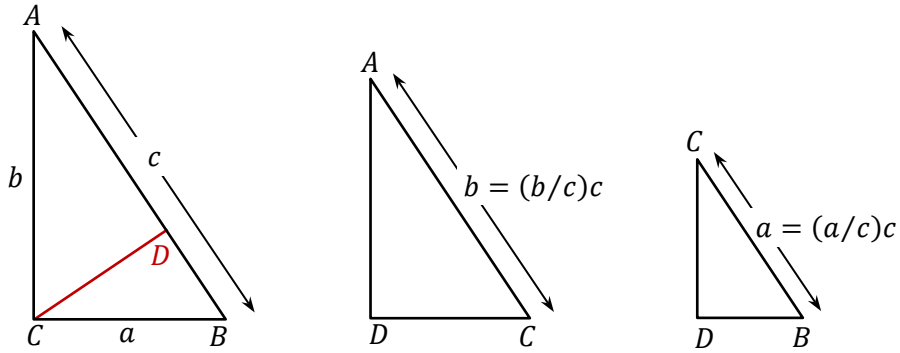
5 pts **Problem 1**) For all values of z inside the unit circle centered at the origin of the complex plane (that is, $|z| < 1$), prove the following identity:

$$(1+z)(1+z^2)(1+z^4)\cdots(1+z^{2^n})\cdots = 1/(1-z).$$

Problem 2) Consider the right-angle triangle ABC, having its right angle at C and side-lengths a, b, and c, as shown.



4 pts a) Dropping the perpendicular CD from the vertex C onto the opposite side AB, show that the right-angle triangles ABC, ACD, and CBD are similar to each other.

4 pts b) Prove the Pythagorean theorem ($a^2 + b^2 = c^2$) by using the fact that ACD is a scaled-down version of ABC in the ratio of $b{:}\,c$, that CBD is similarly a scaled-down version of ABC in the ratio of $a{:}\,c$, and that the triangle areas scale as the square of the ratios of their sides.

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4 pts **Problem 3**) a) Find a compact formula for the function $f(x)$ defined below as an infinite sum:

$$f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}.$$

2 pts b) Assuming that $x > 0$, pick the unique solution (out of two possible solutions found in part a) that would yield the correct (i.e., positive) value for $f(x)$.

2 pts c) Which of the two solutions found in part (a) provides the correct answer in the case of $x = 0$?

2 pts d) For $x < 0$, one must choose between two possible (complex) values for each square root. Explore the range of acceptable solutions for $f(x)$ when $x < 0$, paying particular attention to the transition point at $x = -¼$.

2 pts e) How should one interpret $f(x)$ when x is complex-valued?

Problem 4) The curvilinear uv coordinate system is defined within the Cartesian xy coordinates such that the contours of constant u are ellipses $(x/a)^2 + (y/b)^2 = u$, while the contours of constant v are hyperbolas $(x/c)^2 - (y/d)^2 = v$. Here a, b, c, d are real-valued and positive constants. Within each of the four quadrants of the xy-plane, any given point (x, y) is uniquely associated with a point (u, v).

5 pts a) Identify the range of values of u and v that give a unique value for the corresponding x and y within each of the four quadrants of the xy-plane.

5 pts b) Find the Jacobian of the transformation from the xy to the uv coordinate system.
