**Opti 503A** **Midterm Exam** (2/27/2019) **Time: 75 minutes**

**Please write your name and ID number on all the pages, then staple them together.**
**Answer all the questions.**

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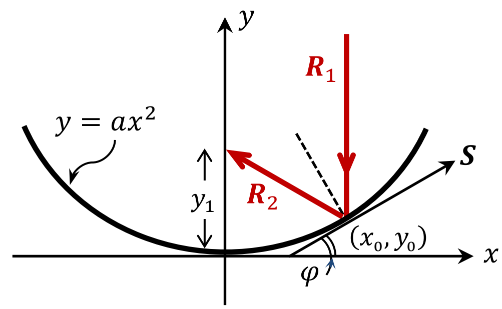
**Problem 1**) The figure shows the cross-sectional profile of a paraboloidal mirror in the $xy$-plane, whose shape is described by the function $y = ax^2$, with $a$ being a positive real constant. A ray of light propagating along the unit-vector $\boldsymbol{R}_1 = -\hat{\boldsymbol{y}}$ arrives at the point $(x_0, y_0)$ on the surface of the mirror. Also shown in the figure is the unit-vector $\boldsymbol{S}$ that is tangent to the parabola at the point of incidence. The reflected ray propagates along the unit-vector $\boldsymbol{R}_2$, crossing the axis of the paraboloid at a height $y_1$ above the mirror's vertex.




4 pts a) Write expressions for the unit-vectors $\boldsymbol{S}$ and $\boldsymbol{R}_2$ in terms of the parameters $a$, $x_0$, $y_0$, and $y_1$.

4 pts b) Considering that $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ make equal angles with the tangent vector $\boldsymbol{S}$, use the vector identity $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = |\boldsymbol{v}_1||\boldsymbol{v}_2| \cos\theta$ to write an equation relating the various system parameters.

4 pts c) Solve the equation found in part (b) to determine the height $y_1$ of the reflected ray $\boldsymbol{R}_2$ on the axis of the paraboloidal mirror. (You will find that $y_1$ does *not* depend on the coordinates $(x_0, y_0)$ of incidence. You have thus obtained the location $y_1$ of the focal point of the mirror as a function of the parameter $a$, which is related to the curvature of the mirror's surface.)

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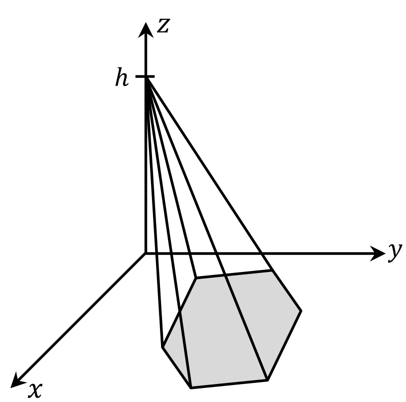
**Problem 2**) The figure shows a polygon in the $xy$-plane, whose sides and corners are connected by straight lines to the point $z = h$ on the $z$-axis. The solid object thus formed is a pyramid whose base area is $A$ and whose height is $h$.




5 pts a) Find a formula for the volume $V$ of the pyramid in terms of $A$ and $h$.

5 pts b) How does the above formula for the volume $V$ change if the pyramid is truncated at the top, by a plane that is parallel to the $xy$-plane and cuts through the pyramid at a height of $z = \alpha h$. Here $\alpha$ is an arbitrary number between 0 and 1.

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5 pts **Problem 3**) Show that the Taylor series expansion of $f(x) = (1 + x)e^x$ is given by

$$\sum_{n=0}^{\infty}[(n+1)/n!]x^n.$$

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5 pts **Problem 4**) a) Let $S_N^{(m)}$ represent the sum of the $m^{\text{th}}$ power of integers from 1 to $N$, that is, $S_N^{(m)} = \sum_{n=1}^{N} n^m$. Using the fact that $\int_{x=1}^{N+1} x^{k-1}\mathrm{d}x = \sum_{n=1}^{N}\int_{x=n}^{n+1} x^{k-1}\mathrm{d}x$ for any arbitrary integer $k \geq 1$, show that the various $S_N^{(m)}$ satisfy the following recursion relation:

$$\sum_{m=0}^{k-1}\binom{k}{m} S_N^{(m)} = (N+1)^k - 1.$$

3 pts b) Starting with $k = 1$, then repeating the same procedure for $k = 2, 3, 4, \cdots$, find closed form expressions for $S_N^{(0)}, S_N^{(1)}, S_N^{(2)}, S_N^{(3)}, \cdots$.

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