**Opti 503A** **Midterm Exam** (2/26/2018) **Time: 75 minutes**

**Please write your name and ID number on all the pages, then staple them together.**
**Answer all the questions.**

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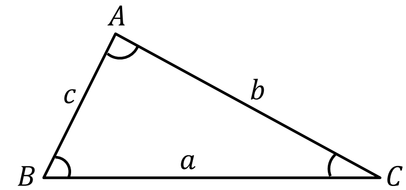
15 pts **Problem 1**) a) Show that the area of the $ABC$ triangle is one-half the product of the lengths of any two sides times the sine of the angle between those sides, that is,




Area of $ABC = ½ab \sin C = ½bc \sin A = ½ac \sin B$.

10 pts b) Use the result of part (a) to prove the following identity for any triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

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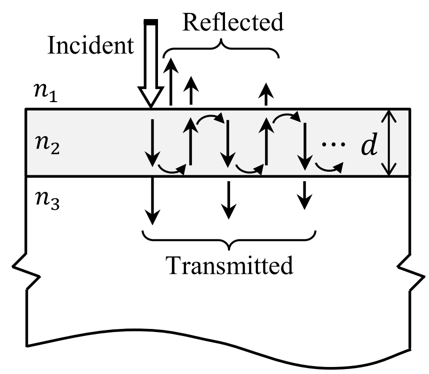
**Problem 2**) A good way to treat monochromatic geometric-optical rays is by assigning a complex amplitude $A = |A| \exp(\mathrm{i}\varphi)$ to each ray, where $|A|$ and $\varphi$ are the magnitude and phase of the ray, respectively. (Monochromatic means "single-color," which refers to the fact that the ray, when propagating in vacuum, is associated with a single wavelength $\lambda_0$.) The refractive indices of optical materials may also be represented by complex numbers, say, $n = n' + \mathrm{i}n''$, where $n'$ is commonly referred to as the refractive index, and $n''$ as the absorption coefficient. (In general, $n'$ and $n''$ are lumped together as a complex number $n$, referred to as the "complex" refractive index.)

When a ray propagates a distance $d$ within a homogeneous medium of (complex) refractive index $n$, its (complex) amplitude $A$ gets multiplied by $\eta = \exp(\mathrm{i}2\pi nd/\lambda_0)$. At the interface between two media, say, medium 1 and medium 2, the ray is partially reflected and partially transmitted. Denoting the (complex) reflection coefficient by $\rho_{12}$ and the (complex) transmission coefficient by $\tau_{12}$, the incident ray amplitude $A$ (arriving from the incidence medium 1 at the interface with medium 2), splits into a reflected ray $\rho_{12}A$ and a transmitted ray $\tau_{12}A$.

20 pts a) With reference to the figure, let a ray of amplitude $A$ and vacuum wavelength $\lambda_0$ arrive from the incidence medium of refractive index $n_1$ at a thin layer of thickness $d$ and refractive index $n_2$. The layer is coated atop a homogeneous, semi-infinite medium of refractive index $n_3$. In terms of the parameters $\rho_{12}, \tau_{12}, \rho_{21}, \tau_{21}, \rho_{23}, \tau_{23}$, and the propagation factor $\eta = \exp(\mathrm{i}2\pi n_2 d/\lambda_0)$, write an infinite series for the total reflected amplitude at the interface between media 1 and 2, and also another infinite series for the transmitted amplitude at the interface between media 2 and 3.




15 pts b) Use the geometric series formula $\sum_{k=0}^{\infty} x^k = 1/(1-x)$, where $|x| < 1$, to find a closed form for the overall reflection and transmission coefficients of the thin layer depicted in the figure.

**Hint**: You are *not* being asked to use the actual formulas for the Fresnel reflection and transmission coefficients at normal incidence, namely, $\rho_{ij} = (n_i - n_j)/(n_i + n_j)$ and $\tau_{ij} = 2n_i/(n_i + n_j)$. Your answers should be expressed in terms of $\rho_{12}, \tau_{12}, \rho_{21}, \tau_{21}, \rho_{23}, \tau_{23}$, and $\eta$.

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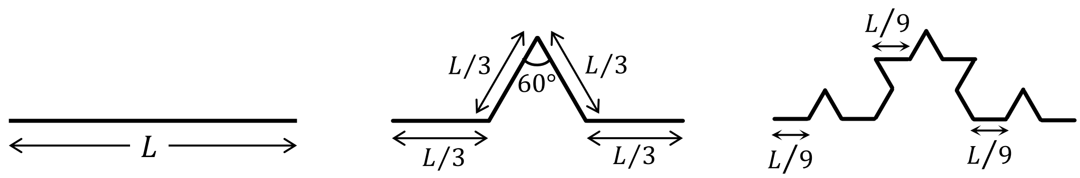
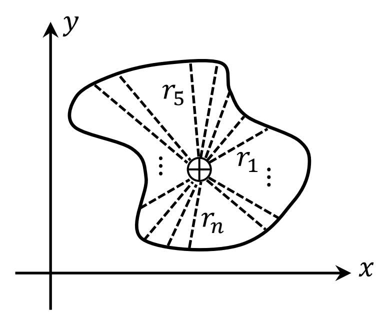
20 pts **Problem 3**) Consider a straight line-segment of length $L$. Take out the middle third of the line, then replace it with two equal-length segments at a $60°$ angle, as shown in the figure. The total length of the line after this first step will be $4L/3$. Repeat the process with each of the four line-segments, so that the overall length of the (broken) line after the second step becomes $16L/9$. Find a formula for the overall length of the broken line after $n$ repeats of the same procedure.




20 pts **Problem 4**) A flat, closed curve in the $xy$-plane is defined by its distance $r(\theta)$ from a central point, where $0 \le \theta \le 2\pi$ is the angular position on the curve, measured relative to some arbitrary reference direction. The area enclosed by the curve is $A = ½\int_0^{2\pi} r^2(\theta)\mathrm{d}\theta$, while the perimeter of the closed curve is $P = \int_0^{2\pi} r(\theta)\mathrm{d}\theta$. Use the method of Lagrange multipliers to find the particular function $r(\theta)$ that maximizes the area $A$ of the closed curve for a given value of its perimeter $P = P_0$.




**Hint**: You may want to work with a discrete version of $r(\theta)$, where the radii $r_1, r_2, r_3, \cdots, r_n, \cdots, r_N$ evenly divide the available range of $\theta$, creating an angular separation of $\Delta\theta$ between adjacent radii. The area and the perimeter of the closed curve may then be approximated as $A = ½\sum r_n^2 \Delta\theta$ and $P = \sum r_n \Delta\theta$.