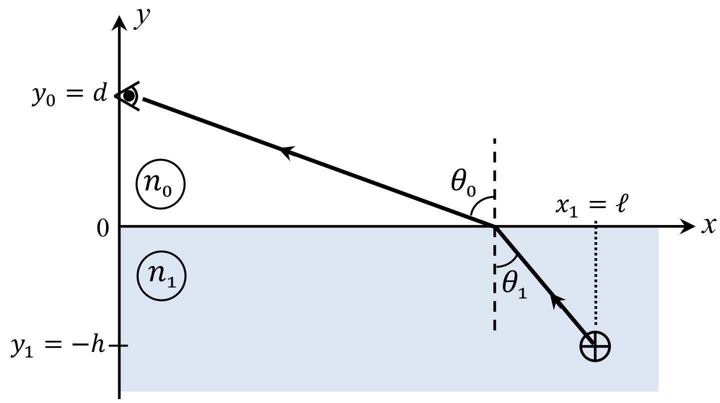
Opti 503A **Midterm Exam** (3/1/2017) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Problem 1) A ray of light travels from an underwater object located at $(x_1, y_1, z_1) = (\ell, -h, 0)$ to an observer's eye at $(x_0, y_0, z_0) = (0, d, 0)$, as shown. The refractive index of the air, where the observer is located, is n_0, while that of the water is n_1. (The speed of light in a medium of refractive index n is given by $v = c/n$, where c is the speed of light in vacuum.)

15 pts a) Locate the x-coordinate of the light ray where it emerges from the water in such a way as to *minimize* the time of travel from the object to the observer's eye.

Note: The answer is a root of a quartic (i.e., 4th degree) polynomial equation. Although such equations are generally solvable, you are being asked here only to find the equation — not its solution.

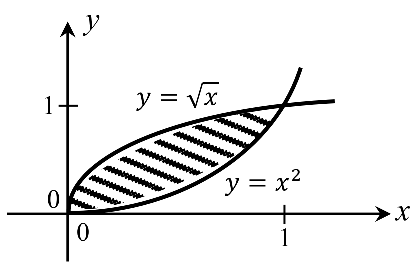


10 pts b) Use the result obtained in part (a) to verify Snell's law of refraction, namely, $n_1 \sin\theta_1 = n_0 \sin\theta_0$.

Problem 2) A transformation from the Cartesian xy coordinate system to the curvilinear uv system is defined by the equations $u = y/x^2$ and $v = x/y^2$.

5 pts a) Within the xy-plane, plot a set of curves that show the contours of constant u, and another set that shows the contours of constant v. Aside from the singularity of the x-axis, where $y = 0$, and the singularity of the y-axis, where $x = 0$, show that the transformation uniquely associates every point (x, y) with a point (u, v), and vice-versa.

10 pts b) Determine the Jacobian of the transformation from xy to uv, and also the Jacobian of the inverse transformation from uv to xy. Confirm that $\partial(x, y)/\partial(u, v)$ is the inverse of $\partial(u, v)/\partial(x, y)$.



10 pts c) Find the area of the region of the xy-plane that is bounded by the curves $y = x^2$ and $y = \sqrt{x}$. (This is the shaded area in the above figure.) Evaluate the relevant integral in both xy and uv coordinate systems, and confirm that the results are identical.

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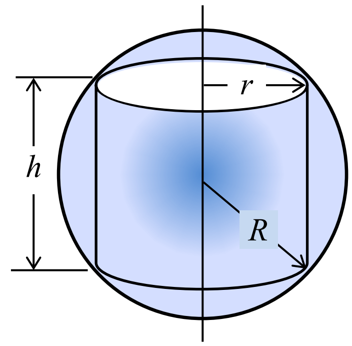
25 pts **Problem 3**) A cylindrical hole of radius r and height h is perforated within a solid sphere of radius R. Use the method of Lagrange multipliers to *maximize* the volume of the cylinder subject to the constraint that its round edges touch the surface of the sphere.



Problem 4) The function $f(x) = \frac{x}{\ln(x+1)}$ is defined for the real variable x where $x \geq -1$.

10 pts a) Use the Taylor series expansion of $\ln(x+1)$ around the point $x = 0$ to estimate the values of $f(x)$ at $x = 0$ and $x = -1$.

10 pts b) Find a recursion relation for the coefficients a_n of the Taylor series expansion of $f(x)$ around $x = 0$, namely, $f(x) = \sum_{n=0}^{\infty} a_n x^n$. What are the numerical values of a_0, a_1, a_2, a_3 and a_4?

5 pts c) Plot the general shape of the function $f(x)$ versus x for $x \geq -1$. Identify some of the important/interesting features of the curve.

Hint: The method needed in part (b) is similar to that used for obtaining the Bernoulli numbers.
