Opti 503A **Midterm Exam** (3/2/2016) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

---

15 pts **Problem 1**) While $(a^b)^c = a^{bc}$, the standard interpretation of a^{b^c} is $a^{(b^c)}$. Given this interpretation, use the chain rule of differentiation to find the derivative $f'(x)$ of the function $f(x) = e^{e^{e^x}} = \exp\{\exp[\exp(x)]\}$.

Problem 2) Which of the following statements are true and which are false? You must provide a brief proof or a counterexample in each case.

4 pts a) If x is irrational, then its inverse, $1/x$, is also irrational.

4 pts b) If x is irrational, then its square root, $\sqrt{x}$, is also irrational.

4 pts c) The sum of a rational number and an irrational number is irrational.

4 pts d) The sum of two irrational numbers is irrational.

4 pts e) The product of two irrational numbers is irrational.

---

15 pts **Problem 3)** According to the *fundamental theorem of arithmetic*, any positive integer greater than 1 is either a prime number or can be written as the product of prime numbers in a unique way. For example, $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$. Using this fundamental theorem, prove that, given any positive integer N, its square root $\sqrt{N}$ is either an integer or an irrational number. In other words, unless N happens to be a perfect square, its square root is always going to be irrational.

---

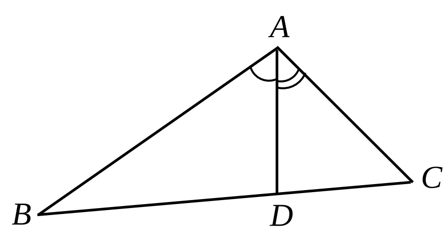
15 pts **Problem 4)** Find the value of the following (infinite) series:

$$S = \frac{1}{1\times5} - \frac{1}{2\times6} + \frac{1}{3\times7} - \frac{1}{4\times8} + \cdots - \frac{(-1)^n}{n\times(n+4)} + \cdots$$

---

15 pts **Problem 5)** Let AD be the bisector of the angle A of the ABC triangle. Show that the ratio $\overline{BD} : \overline{DC}$ is the same as the ratio $\overline{AB} : \overline{AC}$.

Hint: You may want to draw from D a line parallel to AB and another line parallel to AC.



---

20 pts **Problem 6)** Plot the general behavior of the function $f(x) = x^x$ for all real values of x ranging from $-\infty$ to ∞. Identify the point(s) where $f(x)$ has a local maximum or minimum, and pay particular attention to the behavior of the function in the vicinity of $x = 0$.

Hint: For $x < 0$, you may use the identity $\ln x = \ln|x| + \mathrm{i}\pi$.
