Opti 503A **Midterm Exam** (3/4/2015) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

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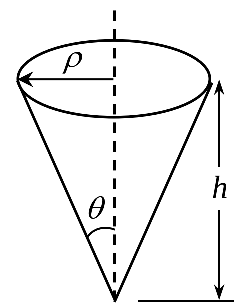
5 pts **Problem 1**) a) Write the Taylor series expansion of the function $f(x) = \exp(x)$ around $x = 0$.

5 pts b) Use the result of part (a) to expand $g(x) = \exp(\mathrm{i}x)$ in a Taylor series around $x = 0$.

5 pts c) What is the value of the infinite sum $S_1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \cdots$?

5 pts d) What is the value of the infinite sum $S_2 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \cdots$?

Problem 2) A paper cup has the shape of a hollow, right-circular cone, with height h, base radius ρ, and cone semi-angle θ, as shown. Clearly, $\rho = h \tan\theta$.

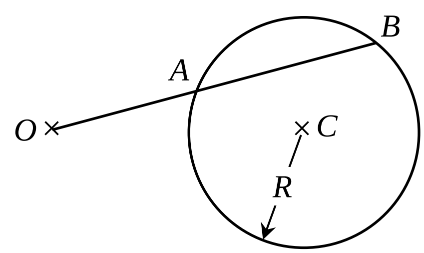


5 pts a) Determine the volume V of the cup as a function of h and θ.

5 pts b) Determine the surface area S of the cup (i.e., area of the flat sheet of paper used to make the cup) as a function of h and θ.

10 pts c) Use the method of Lagrange multipliers to minimize the surface area S subject to the constraint of a fixed volume $V = V_0$.

Problem 3) From an arbitrary point O outside a circle of radius R centered at C, a straight-line OAB is drawn such that it crosses the circle at points A and B, as shown.



10 pts a) Show that the product $\overline{OA} \cdot \overline{OB}$ of the line-segments is the same for all straight lines drawn from O that cross the circle.

6 pts b) Show that $\overline{OA} \cdot \overline{OB} = \overline{OC}^2 - R^2$, where $\overline{OC}$ is the length of the straight-line connecting O to C.

4 pts c) Draw a tangent OD from the external point O to the circle, and verify that $\overline{OD} \cdot \overline{OD} = \overline{OC}^2 - R^2$.

Hint: Use the properties of similar triangles.

Problem 4) The hyperbolic sine and cosine functions are defined as follows:

$$\sinh(x) = ½[\exp(x) - \exp(-x)],$$
$$\cosh(x) = ½[\exp(x) + \exp(-x)].$$

5 pts a) Find the Taylor series expansions of $\sinh(x)$ and $\cosh(x)$ around the point $x = 0$.

5 pts b) Show that, in general, the Taylor series expansion of an even function of x around $x = 0$ contains only *even* powers of x, whereas that of an odd function contains only *odd* powers.

4 pts c) Considering the symmetry of the function $\tanh(x) = \sinh(x)/\cosh(x)$, write the general form of its Taylor series expansion around $x = 0$.

Note: At this point you are *not* being asked to determine the coefficients of the Taylor series, only the *general form* of the series based on the oddness or evenness of the function.

6 pts d) Use the results obtained in parts (a) and (c) to find a recursion relation for the coefficients of the Taylor series of $\tanh(x)$, so that each coefficient a_n can be derived from the preceding coefficients $a_0, a_1, a_2, \dots, a_{n-1}$.

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20 pts **Problem 5**) Let $f(Z) = p(x, y) + \mathrm{i}q(x, y)$ and $g(Z) = r(x, y) + \mathrm{i}s(x, y)$ be complex functions of the complex variable $Z = x + \mathrm{i}y$. Both functions are defined at $Z = Z_0$ and are differentiable at that point. Show that the function $h(Z) = f(Z)g(Z)$ is also differentiable at $Z = Z_0$ by confirming that $h(Z)$ satisfies the Cauchy-Riemann conditions.
