Opti 503A **Midterm Exam** (3/5/2014) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

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4 pts **Problem 1**) a) Plot the function $\ln(1+x)$, which is defined over the semi-infinite interval $x > -1$. Find the Taylor-series expansion of the function around the point $x_0 = 0$.

4 pts b) Plot the function $\ln(1-x)$, which is defined over the semi-infinite interval $x < +1$. Find the Taylor-series expansion of the function around $x_0 = 0$.

4 pts c) Plot the function $\ln\left(\frac{1+x}{1-x}\right)$, which is defined over the interval $-1 < x < +1$. Write the Taylor-series expansion of the function around $x_0 = 0$.

4 pts d) Using the result of part (c) and setting $x = ½$, find an infinite series that converges to ln 3.

4 pts e) Using the result of part (c) and setting $x = ⅓$, find an infinite series that converges to ln 2.

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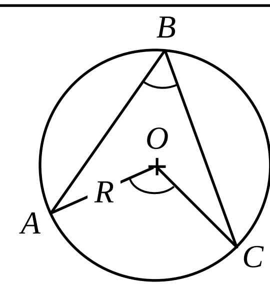
20 pts **Problem 2**) A circle of radius R is centered at the point O. The points A, B, and C are chosen arbitrarily on the perimeter of the circle, and the angles $\widehat{ABC}$ and $\widehat{AOC}$ are formed, as shown in the figure. Show that, in general, $\widehat{ABC} = ½\widehat{AOC}$.



Hint: Draw the diameter of the circle that goes through the point B.

Problem 3) The Gamma function $\Gamma(x)$ is defined for positive values of x as follows:

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy; \qquad x > 0.$$

3 pts a) Show that $\Gamma(1) = 1$.

3 pts b) Show that $\Gamma(2) = 1$.

5 pts c) Use the method of integration by parts to show that $\Gamma(x) = (x-1)\Gamma(x-1)$.

3 pts d) Using the results obtained in parts (a)–(c), argue that $\Gamma(n) = (n-1)!$.

3 pts e) What is the value of $\Gamma(½)$?

3 pts f) What is the value of $\Gamma(n + ½)$, where $n \geq 1$ is an integer?

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20 pts **Problem 4**) The outcome of an experiment is a random integer between 1 and N, each outcome having a probability $p_n \geq 0$. The Shannon entropy H of such an experiment is defined as

$$H(p_1, p_2, \dots, p_N) = -\sum_{n=1}^{N} p_n \ln(p_n).$$

Using the method of Lagrange multipliers, maximize the above entropy subject to the constraint that the sum of the probabilities must be equal to unity, that is, $\sum_{n=1}^{N} p_n = 1$.

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20 pts **Problem 5**) Assuming z_1 and z_2 are arbitrary complex numbers, explain how $z_1^{z_2}$ is calculated.
