Opti 503 **Midterm Exam** (3/2/2011) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

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4 pts **Problem 1**) Show that $\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(2n+1)} = 2\ln 2 - 1.$

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4 pts **Problem 2**) Riemann's zeta function is defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$ where $s > 1$. With the aid of the Bernoulli numbers, it can be shown that $\zeta(4) = \frac{\pi^4}{90}$. Show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}.$

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4 pts **Problem 3**) For a cylindrical can of radius r and height h, whose volume $V(r,h) = \pi r^2 h$ is fixed at V_o, use the method of Lagrange multipliers to determine the minimum surface area $S(r,h) = 2\pi r^2 + 2\pi rh.$

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4 pts **Problem 4**) Use the Cauchy-Riemann conditions to determine if and where the function $f(z) = f(x+\mathrm{i}y) = (x-y)^2 + 2\mathrm{i}(x+y)$ is differentiable. What is the region, if any, within which $f(z)$ is analytic?

Problem 5) The complex function $f(z) = u(r,\theta) + \mathrm{i}\,v(r,\theta)$ is defined in a proper region within the complex-plane $z = r\exp(\mathrm{i}\theta)$.

2 pts a) Derive the Cauchy-Riemann conditions in terms of the partial derivatives of $u(r,\theta)$ and $v(r,\theta)$ with respect to the polar coordinates r and θ.

2 pts b) Use the result in part (a) to show that the function $f(z) = z^{1/2}$ is analytic everywhere in the complex-plane except on a branch-cut.

1 pt c) Identify an appropriate branch-cut for the function $f(z) = z^{1/2}$.

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4 pts **Problem 6**) Use residue calculus to show that

$$\int_0^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}; \qquad a > 0, \quad b > 0.$$
