Opti 403A/503A **Final Exam** (5/12/2021) **Time: 2 hours**

Please write your name and ID number on the first page before scanning/photographing the pages. Answer all the questions.

Problem 1) An integral representation of Bessel functions of the first kind, integer order n, is

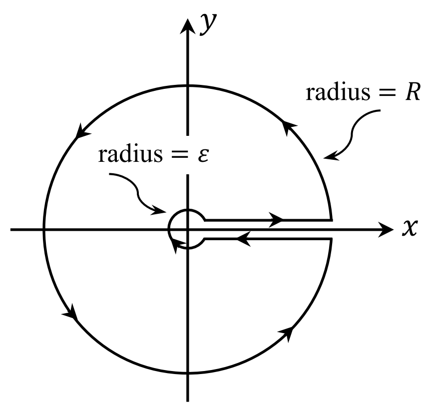
$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}(z\sin\theta - n\theta)}\mathrm{d}\theta, \qquad (n = 0, 1, 2, 3, \cdots), \quad z = \text{arbitrary complex number.}$$

Use this representation to prove the following functional relations among these Bessel functions:

5 pts a) $zJ_{n-1}(z) + zJ_{n+1}(z) = 2nJ_n(z)$.

5 pts b) $J_{n-1}(z) - J_{n+1}(z) = 2\frac{\mathrm{d}}{\mathrm{d}z}J_n(z)$.

Problem 2) The function $f(z) = z^\lambda$, where $\lambda = \lambda' + \mathrm{i}\lambda''$ is an arbitrary complex constant, is defined over the complex z-plane. If the real part λ' of λ happens to be positive or zero, then $f(z) = 0$ at $z = 0$. In contrast, when λ' is negative, $f(z) \to \infty$ as $z \to 0$; in this case the value of the function at $z = 0$ is not specified, and $f(z)$ is said to have a singularity at the origin (i.e., at $z = 0$). Writing $z = |z|e^{\mathrm{i}\varphi}$ and $f(z) = z^\lambda = e^{\lambda \ln z} = \exp[(\lambda' + \mathrm{i}\lambda'')(\ln|z| + \mathrm{i}\varphi)]$, it is evident that a branch-cut is needed to confine the phase angle φ to within a 2π interval — or else every time that φ is incremented by 2π, the function $f(z)$ will acquire a different value. (The only situation in which a branch-cut is *not* needed occurs when $\lambda'' = 0$ and λ' is an integer.) In this problem, we choose to confine the phase angle φ between 0 and 2π (that is, $0 \le \varphi < 2\pi$), so that the positive real axis serves as the branch-cut, as can be inferred from the figure.



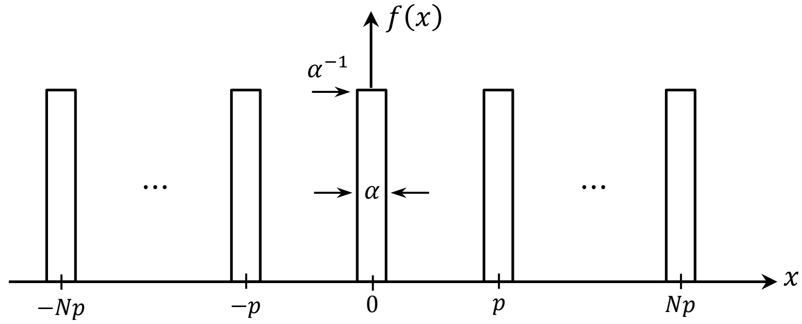
4 pts a) Write expressions for the function $f(z)$ in terms of the various parameters (i.e., ε, R, φ, λ, etc.) when z is located on the large circle of radius R, or on the small circle of radius ε, or on the straight-line-segment that lies immediately above (or immediately below) the real axis between $x = \varepsilon$ and $x = R$.

4 pts b) Evaluate the counterclockwise integral of $f(z)$ around the circle of radius R centered at $z = 0$ in the complex z-plane; that is, find $\oint_{\text{circle}} f(z)\mathrm{d}z$.

4 pts c) Evaluate the clockwise integral of $f(z)$ around the circle of radius ε centered at $z = 0$. Similarly, evaluate the integral of $f(z)$ along the two straight-line-segments that lie immediately above and immediately below the real axis.

3 pts d) In light of the fact that $f(z)$ is analytic throughout the entire region within the closed contour depicted in the figure, use the results obtained in parts (b) and (c) to confirm the validity of the Cauchy-Goursat theorem. (In other words, show that the closed-loop integral is zero.)

Problem 3) The function $f(x)$ consists of $2N+1$ identical rectangular pulses of width α and height α^{-1}, located at regular intervals of p along the x-axis, as shown. In the limit when $\alpha \to 0$ and $N \to \infty$, this function approaches the normalized comb function $p^{-1}\text{comb}(x/p)$. The goal of the present problem is to demonstrate the validity of Parseval's theorem for the comb function, namely, that the area under the square of the function, $|f(x)|^2$, equals the area under the square of its Fourier transform, $|F(s)|^2$ in the limit when $\alpha \to 0$ and $N \to \infty$. This can be achieved in several steps, as follows:



3 pts a) In terms of the parameters N, p, and α, and the special functions $\text{rect}(x)$ and $\text{comb}(x)$, write an expression for the function $f(x)$ shown in the figure.

3 pts b) Find the area under the square of the function; that is, $\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x$.

3 pts c) Find the Fourier transform $F(s)$ of $f(x)$ in terms of the parameters N, p, α, and the special function $\text{sinc}(s)$, which is defined as $\sin(\pi s)/(\pi s)$.

3 pts d) Evaluate the area under the square of $F(s)$, namely, $\int_{-\infty}^{\infty} |F(s)|^2 \mathrm{d}s$. (Ignore the overlap between a $\text{sinc}^2(s)$ function and the tails of its neighboring $\text{sinc}^2(s)$ functions; for sufficiently large N, these functions are fairly narrow and relatively far apart from each other.)

3 pts e) Show that, in the limit when $\alpha \to 0$ and $N \to \infty$, the area obtained in part (d) agrees with that obtained in part (b).

Hint: You may invoke the various theorems of the Fourier transform theory, such as the convolution theorem, multiplication theorem, and scaling theorem. Also feel free to use the fact that $\text{sinc}(s)$ is the Fourier transform of $\text{rect}(x)$, and that $\int_{-\infty}^{\infty} \text{sinc}^2(s)\mathrm{d}s = 1$.

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10 pts **Problem 4**) Use the Cauchy-Goursat theorem of complex analysis (along with other complex-plane techniques) to prove the following identity:

$$\int_{-\infty}^{\infty} \frac{x}{a^2 e^x + b^2 e^{-x}} \mathrm{d}x = \frac{\pi \ln(b/a)}{2ab}, \qquad ab > 0.$$

Hint: Consider using the integration contour shown below.