Opti 503A **Final Exam** (5/13/2020) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

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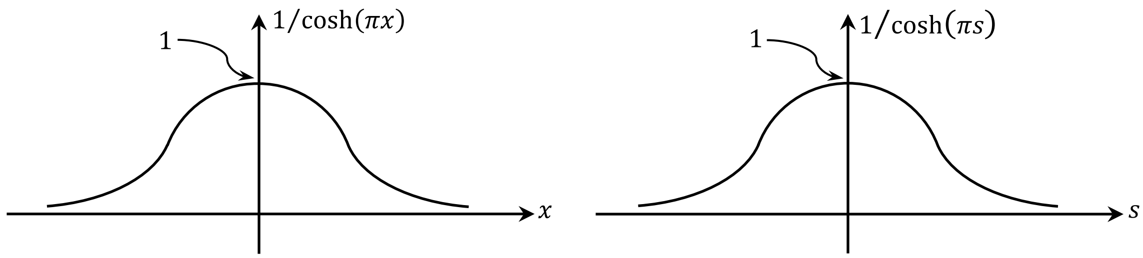
15 pts **Problem 1**) Shown below are plots of $f(x) = 1/\cosh(\pi x)$ and its Fourier transform $F(s) = 1/\cosh(\pi s)$. Using this information in conjunction with some of the Fourier transform theorems, show that the Fourier transform of $g(x) = \delta(x)$ is $G(s) = 1$. Specify the particular theorems as well as other pieces of information that you rely upon in each step of your demonstration.



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15 pts **Problem 2**) The following second-order, linear, ordinary differential equation, in which $n \geq 0$ is an integer and α is an arbitrary parameter, is known as the Laguerre equation:

$$xf''(x) + (1 + \alpha - x)f'(x) + nf(x) = 0.$$

Use the method of Frobenius to find one of the two linearly independent solutions of Laguerre's equation.

Hint: To avoid confusing the parameter n with the summation dummy, assume $f(x) = x^s \sum_{m=0}^{\infty} a_m x^m$. Also, assuming $\alpha \geq 0$ will simplify the derivation.

Problem 3) In this problem, you are asked to derive in two different ways the Fourier transform of the Sign function, namely, $\text{sign}(x) = x/|x|$. The first method, the subject of part (a), is based on a knowledge of the transform of the unit-step function. In part (b), you will start with the function $f(x) = \tanh(\pi x)$, which resembles a smooth version of $\text{sign}(x)$, derive its Fourier transform by complex-plane techniques, then use the fact that $\text{sign}(x) = lim_{\alpha \to 0} \tanh(\pi x/\alpha)$ to arrive at the Fourier transform of $\text{sign}(x)$.

5 pts a) Use the Fourier transform of the Step function, namely, $\mathcal{F}\{\text{step}(x)\} = ½\delta(s) - \mathrm{i}/(2\pi s)$, in conjunction with some of the Fourier transform theorems, to determine the Fourier transform of $\text{sign}(x)$. In each step of your derivation, identify the specific property or theorem that is being relied upon.

10 pts b) Find $\mathcal{F}\{\tanh(\pi x)\} = \int_{-\infty}^{\infty} \tanh(\pi x)\, e^{-\mathrm{i}2\pi s x} \mathrm{d}x$ using Cauchy's theorem and the complex-plane contour of integration shown on the next page. In the diagram, the height of the contour, $N \geq 1$, is an arbitrary integer. You must first argue that, in the limit when $L \to \infty$, the contribution to the loop integral by the vertical legs of the contour is negligible. You will then identify the poles $z_1, z_2, z_3, \cdots, z_N$ of the integrand, and evaluate the residues at these poles. In the final step, you will relate the Fourier transform of $\tanh(\pi x)$ to the sum of the residues at the N poles contained within the closed loop. If you do everything right, you will find that $\mathcal{F}\{\tanh(\pi x)\} = -\mathrm{i}/\sinh(\pi s)$, independently of the value of N.

5 pts c) Let $\alpha > 0$ be an arbitrary real number. Explain why, in the limit when $\alpha \to 0$, the scaled function $\tanh(\pi x/\alpha)$ approaches $\text{sign}(x)$. Use the scaling theorem of Fourier transformation to find the transform of $\tanh(\pi x/\alpha)$, then show that, in the limit when $\alpha \to 0$, this Fourier transform approaches the same function that you found for $\mathcal{F}\{\text{sign}(x)\}$ in part (a).