Opti 503A **Final Exam** (5/8/2019) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

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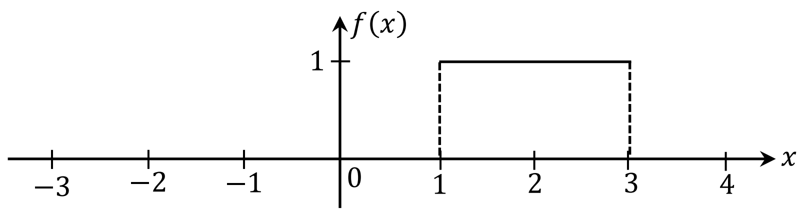
10 pts **Problem 1**) a) Invoking the Cauchy-Riemann conditions, demonstrate that $f(z) = \exp(-z^2)$ is analytic throughout the entire complex z-plane.

5 pts b) Find the derivative $f'(z)$ of $f(z)$ at the arbitrarily chosen point $z = z_0$.

Problem 2) The function $f(x)$ equals 1.0 when $1 \le x \le 3$, and 0.0 otherwise, as shown.



5 pts a) Find the Fourier transform $F(s)$ of $f(x)$ by direct integration.

5 pts b) Express $f(x)$ in terms of the elementary function $\text{rect}(x)$.

5 pts c) Use the shift and scaling theorems of the Fourier transform theory to determine $F(s)$ for the function obtained in part (b). Confirm that your result agrees with that obtained in part (a).

Problem 3) The cross-correlation between the functions $f(x)$ and $g(x)$ is defined as follows:

$$f(x) \otimes g(x) = \int_{-\infty}^{\infty} f(x')g(x' - x)\mathrm{d}x'.$$

5 pts a) In what way does cross-correlation differ from the convolution operation?

5 pts b) Denoting the Fourier transforms of $f(x)$ and $g(x)$ by $F(s)$ and $G(s)$, respectively, show that the Fourier transform of the cross-correlation function is given by

$$\mathcal{F}\{f(x) \otimes g(x)\} = F(s)G(-s).$$

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10 pts **Problem 4**) Use the method of Fourier transformation to solve the following first-order linear ordinary differential equation with constant coefficients:

$$g'(x) + g(x) = \text{rect}(x)\cos(2\pi s_0 x).$$

Note that the excitation function appearing on the right-hand side of the above equation has a constant positive frequency s_0, and that the excitation is limited to the range $-½ \le x \le ½$. Your solution for $g(x)$ must cover the entire range of x from $-\infty$ to ∞.

Hint: $\cos(2\pi s_0 x) = [\exp(\mathrm{i}2\pi s_0 x) + \exp(-\mathrm{i}2\pi s_0 x)]/2$ and $\sin(2\pi s_0 x) = [\exp(\mathrm{i}2\pi s_0 x) - \exp(-\mathrm{i}2\pi s_0 x)]/2\mathrm{i}$. You will need to use the differentiation theorem of Fourier transform theory, and also carry out several integrations in the upper-half as well as lower-half of the complex s-plane.
