Opti 503A **Final Exam** (5/9/2018) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

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10 pts **Problem 1**) Invoking Euler's Gamma function, $\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} \mathrm{d}y$ (defined here for $x > 0$), evaluate the definite integral $\int_0^\infty \exp(-x^3)\, \mathrm{d}x$.

Hint: The evaluated integral will depend on $\Gamma(⅓)$.

Problem 2) The goal of the present problem is to demonstrate that the Fourier transform $F(s)$ of the function $f(x) = \cos(2\pi s_0 x)$ is a pair of delta-functions located at $s = \pm s_0$. Consider the function $f_\alpha(x) = \exp(-\alpha x^2)\cos(2\pi s_0 x)$, where α is a positive real parameter which will eventually be made to approach zero, thus allowing $f_\alpha(x)$ to approach $f(x)$.

10 pts a) Use complex-plane integration methods to determine the Fourier transform of $e^{-\alpha x^2} e^{\mathrm{i}2\pi s_0 x}$.

5 pts b) Repeat part (a) for the function $e^{-\alpha x^2} e^{-\mathrm{i}2\pi s_0 x}$.

5 pts c) Combine the results obtained is parts (a) and (b) to show that the Fourier transform $F_\alpha(s)$ of the function $f_\alpha(x)$ approaches a pair of δ-functions in the limit when $\alpha \to 0$.

Hint: The Fourier transform of $g(x) = \exp(-\pi x^2)$ is $G(s) = \exp(-\pi s^2)$, indicating that $\int_{-\infty}^{\infty} \exp(-\pi x^2)\, \mathrm{d}x = G(0) = 1$.

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20 pts **Problem 3**) Using complex-plane integration methods, evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2-1)(x^2-2\mathrm{i}x-2)}.$$

The contour of integration may be closed in either the upper-half or the lower-half of the complex z-plane. Confirm that the result of integration is independent of the half-plane chosen to close the contour.

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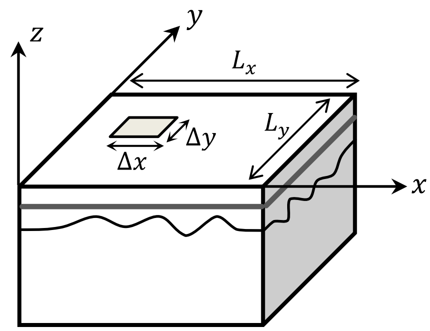
25 pts **Problem 4**) In the method of stationary-phase approximation, the following integral is evaluated by focusing attention at and around stationary points such as x_0, where $g'(x_0) = 0$.

$$\int_a^b f(x) \exp[\mathrm{i}\eta g(x)]\, \mathrm{d}x.$$

Here $f(x)$ is a generally complex-valued function of the real variable x, η is a large (positive or negative) real number, and $g(x)$ is a real-valued function of x. In the class, we assumed that $a < x_0 < b$, and $g''(x_0) \neq 0$, then proceeded to evaluate the contribution of x_0 to the above integral in terms of $f(x_0)$, $g(x_0)$, η, and $g''(x_0)$. In the present problem, you are asked to assume that $g''(x_0) = 0$ but $g'''(x_0) \neq 0$. Applying to method of stationary-phase approximation, estimate the value of the above integral in terms of $f(x_0)$, $g(x_0)$, η, and $g'''(x_0)$.

Hint: $\cos(\pi/6) = \sqrt{3}/2$ and $\int_0^\infty \exp(-x^3)\, \mathrm{d}x = ⅓\Gamma(⅓)$, as shown in Problem 1.

Problem 5) The membrane of a rectangular drumhead is under isotropic tension T [Newton/m] from all sides. The membrane is firmly affixed to the drumhead at three of its four boundaries, namely, at $x = 0, y = 0$, and $x = L_x$, but is free to vibrate (in the z-direction) at the fourth boundary located at $y = L_y$. The membrane's mass-density and friction coefficient (per unit area) are specified as ρ [kg/m²] and β [kg/(m² · sec)], respectively.



5 pts a) Making the usual assumption that the slope of the membrane's displacement $z(x, y, t)$ is small, write the governing equation of motion (in the absence of external forces) in terms of the wave velocity $v = \sqrt{T/\rho}$ and the damping coefficient $\gamma = \beta/\rho$.

5 pts b) Specify the relevant *boundary* conditions and *initial* conditions for the vibrating membrane.

10 pts c) Find all the admissible vibrational *modes* of the membrane by solving the partial differential equation obtained in part (a) subject to the boundary conditions specified in part (b).

5 pts d) Write a general expression for the vibration amplitude $z(x, y, t)$ as a *superposition* of all the various modes obtained in part (c). Explain (in words) how the unknown coefficients of this general expression can be determined.
