Opti 503A **Final Exam** (5/10/2017) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Problem 1) Use the method of integration by parts to verify the following identities:

10 pts a) $\int_0^1 x^n \ln(x)\, \mathrm{d}x = -\frac{1}{(n+1)^2}; \qquad n = 0, 1, 2, 3, \cdots.$

10 pts b) $$\int_0^1 \frac{\ln x}{(1+x)^2}\, \mathrm{d}x = -\ln 2.$$

5 pts c) Make a change of variable to convert the above result, part (b), to the following identity:

$$\int_0^\infty \frac{x}{\cosh^2(x)}\, \mathrm{d}x = \ln 2.$$

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15 pts **Problem 2**) a) Solve the first-order ordinary differential equation $f'(x) + \eta f(x) = \mathrm{Rect}(x)$ using the Fourier transform method. Here η is a real-valued and positive constant.

5 pts b) Verify that your solution is, in fact, continuous at $x = \pm\frac{1}{2}$, where the excitation function $\mathrm{Rect}(x)$ is discontinuous.

5 pts c) Plot the function $f(x)$ over its entire domain $-\infty < x < \infty$.

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20 pts **Problem 3**) Use the complex-plane integration techniques to verify the stated value of the following integral:

$$\int_0^\infty \frac{\cos(\alpha x)}{\cosh(\beta x)}\, \mathrm{d}x = \frac{\pi/2\beta}{\cosh(\pi\alpha/2\beta)}; \qquad \mathrm{Re}(\beta) > 0, \ \ \alpha \text{ any real number.}$$

Hint: Considering that the integrand $\cos(\alpha x)/\cosh(\beta x)$ is an even function of x, the domain of the integral can be extended from $(0, \infty)$ to $(-\infty, \infty)$. Also, the identity $\cos(\alpha x) = \frac{1}{2}[\exp(\mathrm{i}\alpha x) + \exp(-\mathrm{i}\alpha x)]$ may be useful.

Problem 4) Consider the following diffusion equation in one-dimensional space:

$$\alpha \frac{\partial^2}{\partial x^2} f(x,t) = \frac{\partial}{\partial t} f(x,t); \qquad \alpha > 0.$$

The initial condition is given as $f(x, t = 0) = \frac{\mathrm{d}}{\mathrm{d}x} \exp(-\pi x^2) = -2\pi x \exp(-\pi x^2)$. No boundary conditions are specified as the function $f(x,t)$ is allowed to spread along the x-axis with no restrictions other than $f(x,t) \to 0$ as $x \to \pm\infty$. In this problem you are asked to solve the above partial differential equation and find $f(x,t)$ for $t \geq 0$.

10 pts a) Defining the Fourier transform $F(s,t) = \int_{-\infty}^{\infty} f(x,t) \exp(-\mathrm{i}2\pi s x)\, \mathrm{d}x$, write the corresponding partial differential equation for $F(s,t)$. (Note that the transformation is with respect to x, but *not* with respect to t.)

5 pts b) Find the transform of the initial condition, $F(s, t = 0) = \int_{-\infty}^{\infty} f(x, t = 0) \exp(-\mathrm{i}2\pi s x)\, \mathrm{d}x$.

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5 pts c) Solve the partial differential equation obtained in part (a) to find $F(s,t)$ using $F(s,t=0)$ as your initial condition.

10 pts d) Perform an inverse Fourier transformation to arrive at $f(x,t)$ for $t \geq 0$.
