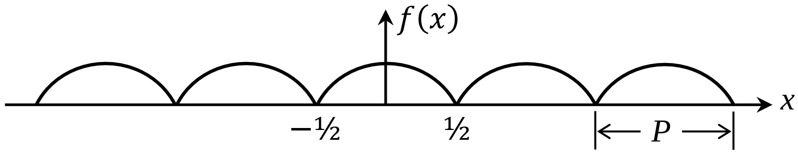
Opti 503A **Final Exam** (5/9/2016) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

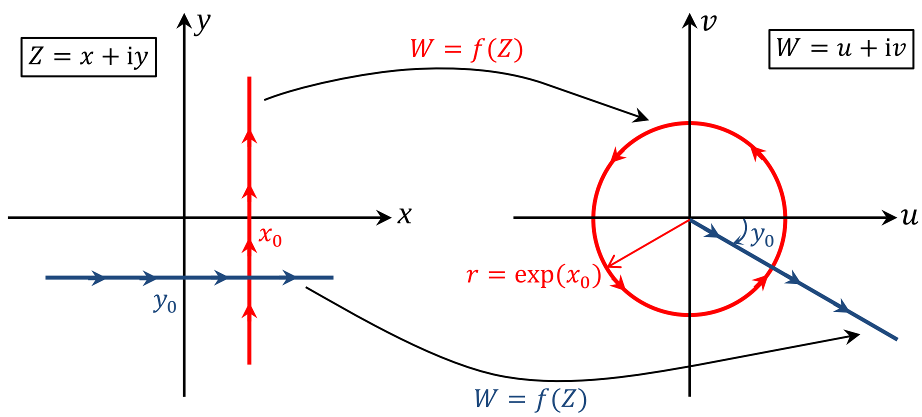
Problem 1) The function $f(x) = |\cos(\pi x)|$ defined on the real x-axis is periodic, with a period $P = 1$, as shown.



10 pts a) Using a combination of the functions $\text{Rect}(x)$, $\text{Comb}(x)$, and $\cos(\pi x)$, express $f(x)$ *without* invoking the absolute value operator $|\cdot|$.

15 pts b) Find the Fourier series expansion of $f(x)$.

Problem 2) A complex function $W = f(Z)$ can be said to project certain contours in the complex Z-plane to the corresponding contours in the complex W-plane. For example, as shown in the figure below, $f(Z) = \exp(Z) = \exp(x + \mathrm{i}y) = \exp(x)\exp(\mathrm{i}y)$ projects straight vertical lines of the Z-plane onto circles of radius $r = \exp(x)$ in the W-plane. Similarly, straight horizontal lines of the Z-plane are projected by $\exp(Z)$ onto straight, semi-infinite lines emanating from the origin of the W-plane, which make an angle $\varphi = y$ with the u-axis.



Draw similar contour projections for the following complex functions:

8 pts a) $f_1(Z) = \ln Z$.

8 pts b) $f_2(Z) = 1/(Z - 1)$.

9 pts c) $f_3(Z) = 1/\sqrt{Z - \mathrm{i}}$.

Problem 3) Hyperbolic sine and cosine functions are defined in the complex z-plane as follows:

$$\cosh(z) = ½[\exp(z) + \exp(-z)]; \qquad \sinh(z) = ½[\exp(z) - \exp(-z)].$$

5 pts a) Writing $z = x + \mathrm{i}y$, show that $\cosh(z) = \cosh(x)\cos(y) + \mathrm{i}\sinh(x)\sin(y)$.

5 pts b) Find all the zeros of $\cosh(z)$, namely, all the points in the z-plane where $\cosh(z) = 0$.

5 pts c) On the horizontal line $z = x + \mathrm{i}\pi$, show that $\cosh(z) = \cosh(x + \mathrm{i}\pi) = -\cosh(x)$.

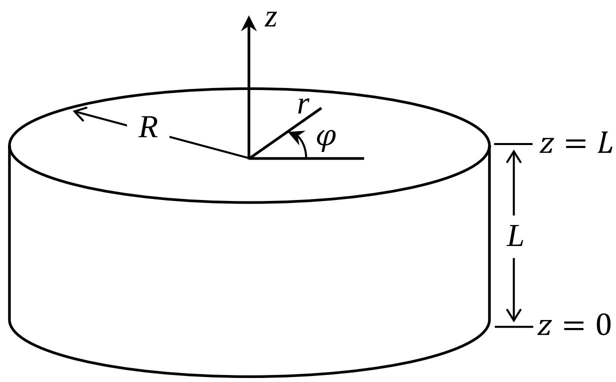
10 pts d) Using complex-plane techniques based on the Cauchy-Goursat theorem, prove the validity of the following definite integrals:

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\cosh(x)} dx = \pi.$ (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{\cosh(x)} dx = 0.$ (iii) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{\cosh(x)} dx = \pi^3/4.$

Problem 4) In the cylindrical coordinates system (r, φ, z), the (scalar) wave equation in three-dimensional space is written as follows:

$$v^2\nabla^2\psi(\boldsymbol{r}, t) = v^2\left(\frac{\partial^2\psi}{\partial r^2} + \frac{\partial\psi}{r\partial r} + \frac{\partial^2\psi}{r^2\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}\right) = \frac{\partial^2\psi}{\partial t^2} + \gamma\frac{\partial\psi}{\partial t}.$$

Here ψ represents the vibration amplitude at location $\boldsymbol{r} = (r, \varphi, z)$ in space and instant t in time, v is the velocity of wave propagation in space, and γ is the damping coefficient.



In the system depicted in the above figure, the wave is confined to the interior of a hollow cylindrical can of radius R and length L, where the boundary conditions are given by

$$\psi(r = R, \varphi, z, t) = 0; \qquad \psi(r, \varphi, z = 0, t) = 0; \qquad \psi(r, \varphi, z = L, t) = 0.$$

25 pts Use the method of separation of variables to determine the general form of the *real-valued* wave function $\psi(\boldsymbol{r}, t)$ inside the can. As usual, you may use ρ_{mn} to represent the n^{th} zero of $J_m(\rho)$, a Bessel function of the first kind, order m. You may also use the positive integer ℓ to represent the mode index for the z-dependent part of the wave function. As for the time-dependent part, allow for the possibility of over-damping, critical damping, and under-damping.
