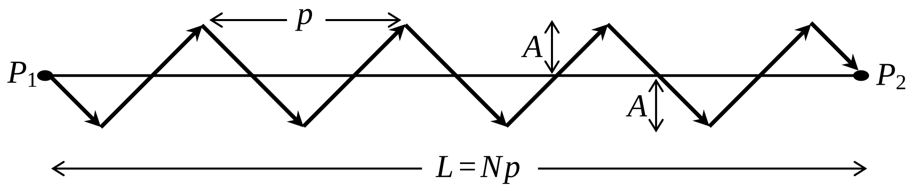
Opti 503A **Final Exam** (5/12/2015) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Problem 1) Shown below is a drunkard's zigzag path as he walks from P_1 to P_2 — two points that are separated by a distance L along a straight line. The path is a periodic saw-tooth function with amplitude A and period $p = L/N$, where N is an integer.



5 pts a) Write a formula for the total distance D travelled between P_1 and P_2 along the zigzag path.

5 pts b) What is the limit of D when $A \to 0$ provided that N is kept constant?

5 pts c) What is the limit of D when $A \to 0$ while $N \to \infty$ in such a way as to keep the product NA constant?

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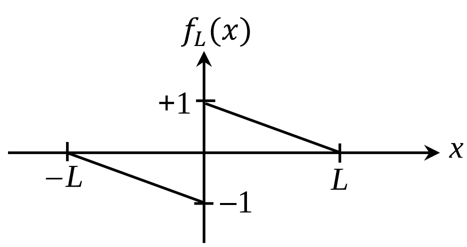
10 pts **Problem 2**) a) Find the Fourier transform $F_L(s)$ of the function $f_L(x)$ depicted below.

(**Hint**: Use integration by parts.)



$$f_L(x) = \begin{cases} -1 - (x/L), & -L \le x < 0; \\ 1 - (x/L), & 0 < x \le L. \end{cases}$$

2 pts b) The function $f_L(x)$ approaches a well-known, standard function in the limit when $L \to \infty$. Identify that function.

3 pts c) Determine the Fourier transform of the standard function which you identified in part (b) by finding the limit when $L \to \infty$ of $F_L(s)$ obtained in part (a).

10 pts d) Find the inverse Fourier transform of $F(s) = 1/(\mathrm{i}\pi s)$ using complex-plane integration. You must specify the contours of integration for both $x > 0$ and $x < 0$, and also explain why certain parts of the contours do not contribute to the inverse Fourier integral.

Problem 3) An ordinary, 2nd order, equi-dimensional, homogeneous differential equation with coefficients a, b, and c (which are arbitrary constants) is written as follows:

$$ax^2 f''(x) + bxf'(x) + cf(x) = 0.$$

5 pts a) Find a solution $f(x)$ of the above equation.

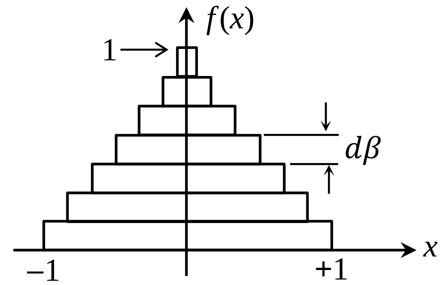
5 pts b) Under what circumstances does the method that you employed to solve the above equation yield only one solution instead of two independent solutions?

5 pts c) In general, when $s_1 \neq s_2$, the solution may be written as $f(x) = A_1 x^{s_1} + A_2 x^{s_2}$, where A_1, A_2 are arbitrary constants. Suppose that $A_1 = A_0/(s_1 - s_2)$ and $A_2 = -A_0/(s_1 - s_2)$. What would become of this particular solution, $f(x)$, in the limit when $s_1 \to s_2$? (**Hint**: $x^s = e^{s \ln x}$)

5 pts d) Confirm that the solution that you found in part (c) above does in fact satisfy the differential equation in the special case when $b = a \pm 2\sqrt{ac}$.

Problem 4) A superposition of a large number of rectangular functions of variable width β and infinitesimal height $d\beta$ produces the following function:

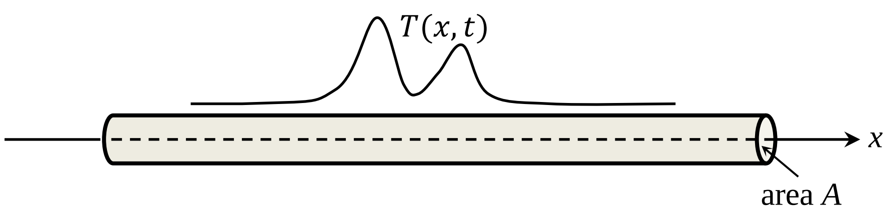
$$f(x) = \tfrac{1}{2} \int_0^2 \text{Rect}(x/\beta)\, d\beta.$$



5 pts a) What well-known, standard function does $f(x)$ represent?

10 pts b) Find the Fourier transform $F(s)$ of $f(x)$, and confirm that $F(s)$ does in fact correspond to the standard function which you associated with $f(x)$ in part (a).

Problem 5) An infinitely long, thin, straight, uniform rod has an initial temperature distribution $T(x, t = 0) = T_0(x)$. In this problem, you are asked to use the method of separation of variables to determine the temperature profile $T(x, t)$ for all times $t \geq 0$ in order to confirm the solution to the same problem that was obtained in the class using Fourier transformation.



5 pts a) Write the one-dimensional heat diffusion equation for $T(x, t)$ assuming no heat loss to the environment. Identify the material parameter(s) that appear in the equation by their names and their units.

10 pts b) Use the method of separation of variables to solve the above diffusion equation and obtain the general form of its separable solutions. Explain why it is necessary for the separation constant to be negative.

10 pts c) Express the complete solution to the problem as a superposition of the various separable solutions obtained in part (b). Show that your solution is identical to the one obtained in the class by Fourier transformation.
