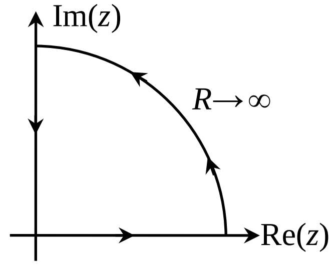
Opti 503A **Final Exam** (5/7/2013) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

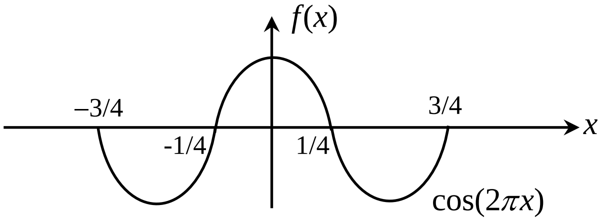
---

10 pts **Problem 1**) Using the contour shown below, verify that $\int_0^{\infty} \frac{\mathrm{d}x}{x^4 + 4a^4} = \frac{\pi}{8a^3}$. Here a is a positive, real-valued parameter.



---

10 pts **Problem 2**) Use two different methods to determine the Fourier transform $F(s)$ of the function $f(x)$ shown below. The function is equal to $\cos(2\pi x)$ when $|x| < 3/4$, and zero otherwise. The first method is direct integration. For the second method, express $f(x)$ as the product of a periodic function and a rectangular pulse, then use the convolution theorem. Confirm that the two methods yield precisely the same result.



---

10 pts **Problem 3**) Use the method of Frobenius to find a solution to the following differential equation:

$$x^2 f''(x) + x f'(x) - (x^2 + p^2) f(x) = 0.$$

The constant p is real-valued and positive.

Note: The equation differs from Bessel's equation only in the sign of x^2 in the coefficient of $f(x)$.

---

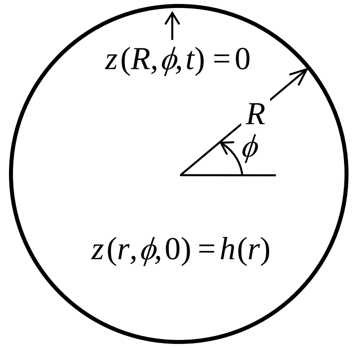
10 pts **Problem 4**) The real-valued function $h(x)$ is defined over the interval $0 \le x \le L$. It is desired to expand this function in a Fourier cosine series, as follows:

$$h(x) = \sum_{n=0}^{\infty} c_n \cos(\pi n x / L).$$

Find the coefficients c_n.

Problem 5) A thin membrane of radius R vibrates with an amplitude $z(r, \phi, t)$. Ignoring friction losses and denoting by v the velocity of wave propagation in the membrane, the governing equation is found to be

$$v^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)z(r,\phi,t)=\frac{\partial^2 z(r,\phi,t)}{\partial t^2}.$$



The membrane is firmly attached to a ring of radius R, so that $z(R,\phi,t)=0$. The initial amplitude of the vibrations is independent of the azimuthal coordinate ϕ, and is given by $z(r,\phi,0)=h(r)$. The membrane's initial velocity is zero, that is, $\partial z(r,\phi,t)/\partial t\big|_{t=0}=0$. The vibration amplitude for all times t is thus given by

$$z(r,\phi,t)=\sum_{n=1}^{\infty}c_n J_0(r_{0n}r/R)\cos(\omega_n t),$$

where r_{0n} is the n^{th} zero of $J_0(r)$, the Bessel function of first kind, 0^{th} order.

2 pts a) Verify that the initial velocity of the membrane at $t=0$ is in fact zero everywhere.

3 pts b) What is the value of ω_n in terms of r_{0n}, R and v?

5 pts c) Find the values of c_n in terms of $h(r)$ and other known parameters of the system.