Opti 503 **Final Exam** (5/10/2011) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

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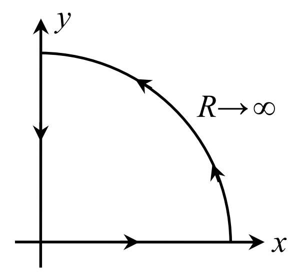
8 pts **Problem 1**) Using the contour shown, verify that $\int_0^\infty \frac{x\,\mathrm{d}x}{x^4+1} = \frac{\pi}{4}$.



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8 pts **Problem 2**) Using the method of Frobenius, find the solution to the following differential equation:

$$x^2 \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + x\frac{\mathrm{d}f(x)}{\mathrm{d}x} + (x^2 - \tfrac{1}{4}) f(x) = 0.$$

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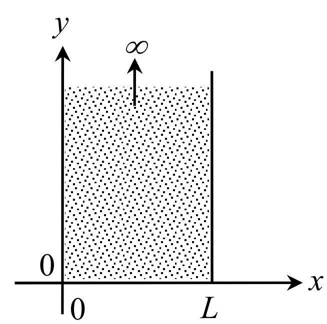
8 pts **Problem 3**) A thin strip of a homogeneous material is located in the xy-plane, in the region $0 \le x \le L$ and $0 \le y < \infty$. The slab is isolated from the environment on all sides, except at its lower edge along the x-axis, where the temperature is maintained at $T(x, y=0) = T_\mathrm{o}(x)$ over the entire interval $0 \le x \le L$. Considering that the steady-state temperature distribution satisfies the 2D Laplace's equation

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0,$$

determine the temperature profile $T(x, y)$ of the slab.



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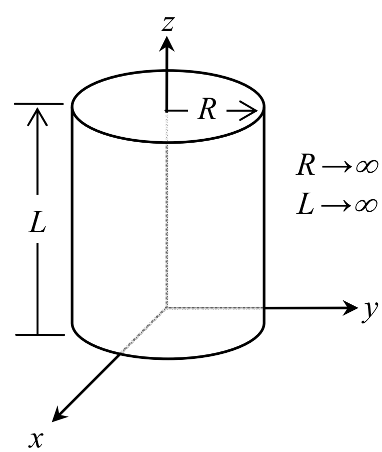
8 pts **Problem 4**) A large, solid, homogeneous cylinder of height L and radius R sits atop the xy-plane and is centered on the z-axis. (For all practical purposes you may assume that R and L approach infinity.) The cylinder is isolated on all sides except on its bottom facet, where the temperature is maintained at $T(r, \phi, z=0) = T_\mathrm{o}(r)$. The steady-state temperature distribution, being independent of the azimuthal angle ϕ, satisfies the 2D Laplace's equation in cylindrical coordinates, namely,

$$\frac{\partial^2 T(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,z)}{\partial r} + \frac{\partial^2 T(r,z)}{\partial z^2} = 0.$$

Find the steady-state temperature profile $T(r, z)$ of the cylinder.



Problem 5) Consider the second-order linear differential equation

$$a_0(x)\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x} + [a_2(x) + \lambda a_3(x)]f(x) = 0.$$

4 pts a) Multiply the equation with an arbitrary function $g(x)$, then determine $g(x)$ such that the equation may be written in the standard Sturm-Liouville form, namely,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right] + [q(x) + \lambda r(x)]f(x) = 0.$$

Derive explicit expressions for $p(x)$, $q(x)$, and $r(x)$ in terms of $a_0(x)$, $a_1(x)$, $a_2(x)$ and $a_3(x)$.

4 pts b) Reduce each of the following equations to the standard Sturm-Liouville form:

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + \mathrm{cotg}(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x} + \lambda f(x) = 0.$$

$$x\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + (1-x)\frac{\mathrm{d}f(x)}{\mathrm{d}x} + (x^2 + \lambda)f(x) = 0.$$
