Problem 13) Method 1: Let $\varepsilon = 1/365$. The first person (A) has a certain birthday. The chance that the second person (B) will have a different birthday is $(1-\varepsilon)$. The chance that the third person (C) will have a different birthday than the first two is $(1-2\varepsilon)$. Thus, the chance that all n people in the room have different birthdays is

$$\text{Probability of no coincident birthdays} = (1-\varepsilon)(1-2\varepsilon)\cdots[1-(n-1)\varepsilon].$$

For an integer m, if $m\varepsilon$ is sufficiently small, one can approximate $(1-m\varepsilon)$ with $(1-\varepsilon)^m$, in which case the above probability becomes (approximately):

$$\begin{aligned}\text{Probability of no coincident birthdays} &\cong (1-\varepsilon)(1-\varepsilon)^2\cdots(1-\varepsilon)^{n-1}\\ &= (1-\varepsilon)^{1+2+\cdots+(n-1)}\\ &= (1-\varepsilon)^{n(n-1)/2}\\ &= (1-\varepsilon)^{\binom{n}{2}}.\end{aligned}$$

For $n = 25$, there will be no coincident birthdays with a probability of $(364/365)^{300} = 0.44$, which means that there is 56% chance that at least two out of the 25 will have the same birthday. Alternatively, given that for sufficiently small values of $m\varepsilon$, $\ln(1-m\varepsilon) \cong -m\varepsilon$, we will have

$$\ln\{(1-\varepsilon)(1-2\varepsilon)\cdots[1-(n-1)\varepsilon]\} \cong -[1+2+\cdots+(n-1)]\varepsilon = -½n(n-1)\varepsilon.$$

Consequently, the probability of no coincident birthdays is about $e^{-½n(n-1)\varepsilon}$, which, for $n = 25$ is, once again, equal to 0.44.

Method 2: The total number of distinct pairs of individuals is $\binom{n}{2}$. The probability that the two individuals in any given pair will have different birthdays is $(1-\varepsilon)$. Therefore, the probability that no pair of individuals in the room will share a birthday is

$$\text{Probability of no coincident birthdays} = (1-\varepsilon)^{\binom{n}{2}}.$$

Discussion: The two methods are seen to yield similar results when $n\varepsilon$ is sufficiently small. Of these, *Method 1* is **exact** and *Method 2* is **approximate**. To see the approximate nature of *Method 2*, consider the case of $n = 3$, and let the three individuals be A, B, and C. The first pair (A, B) will have distinct birthdays with a probability of $(1-\varepsilon)$. Similarly, the second pair (A, C) will have distinct birthdays with a probability of $(1-\varepsilon)$. However, the case of the third pair (B, C) is somewhat different. Since it has already been established that neither B nor C has the same birthday as A, each individual of the third pair must have a birthday on one of the remaining 364 days of the year. Therefore, the probability of having distinct birthdays for the third pair is $363/364$, which no longer equals $(1-\varepsilon)$.