Solution to Problem 12) $H(x) = -\sum_{n=1}^{N} p_n \ln p_n = \sum_{n=1}^{N} p_n \ln(1/p_n)$ is the expected value of the random variable $\zeta = \ln(1/p_n)$. Considering that $\ln \zeta$ is a convex cap function of ζ, the Jensen inequality $\langle \ln \zeta \rangle \leq \ln(\langle \zeta \rangle)$ may be invoked to arrive at

$$H(x) \leq \ln(\langle 1/p_n \rangle) = \ln[\sum_{n=1}^{N} p_n(1/p_n)] = \ln N. \tag{1}$$

It is obvious that the equality will hold if the outcomes of x are equally likely, that is, if $p_n = 1/N$ for all n. Conversely, if the outcomes are not equi-probable, there must exist at least one value of n, say, $n = m$, for which $p_m > 1/N$. We may then write

$$\ln N = \ln \sum_{n=1}^{N} [p_n(1/p_n)] = \ln\{p_m(1/p_m) + (1 - p_m) \sum_{n \neq m} [p_n/(1 - p_m)](1/p_n)\}. \tag{2}$$

Now, since it is assumed that $p_m > 1/N$, we must have

$$\sum_{n \neq m} [p_n/(1 - p_m)](1/p_n) = (N - 1)/(1 - p_m) > N. \tag{3}$$

Therefore, $y_1 = 1/p_m$ and $y_2 = \sum_{n \neq m} [p_n/(1 - p_m)](1/p_n)$, being on opposite sides of $p_m y_1 + (1 - p_m) y_2 = N$, cannot be equal. Given that $\ln y$ is a convex cap function of y, and that y_1 and y_2 are *not* coincident, one can assert the *strict* inequality $\ln N > p_m \ln y_1 + (1 - p_m) \ln y_2$, that is,

$$\ln N > p_m \ln(1/p_m) + (1 - p_m) \ln \sum_{n \neq m} [p_n/(1 - p_m)](1/p_n). \tag{4}$$

Now, $\sum_{n \neq m} [p_n/(1 - p_m)](1/p_n)$ is the expected value of the random variable $1/p_n$ with $n \neq m$ and probability distribution $p_n/(1 - p_m)$. Consequently, Jensen's inequality may be applied one more time to the convex cap function $\ln y$ to arrive at

$$\ln N > p_m \ln(1/p_m) + (1 - p_m) \sum_{n \neq m} [p_n/(1 - p_m)] \ln(1/p_n)$$
$$= \sum_{n=1}^{N} p_n \ln(1/p_n) = H(x). \tag{5}$$

The entropy $H(x)$ of the random variable x, which assumes the values $x_1, x_2, \cdots, x_N$ with a non-uniform probability distribution, is thus seen to remain strictly below $\ln N$.