Solution to Problem 8) a) Poisson's discrete probability density function is represented by a set of equally spaced delta-functions located at $x = 0, 1, 2, \cdots, n, \cdots$. Its Fourier transform is readily evaluated as follows:

$$\begin{aligned}\psi(s) &= \int_{-\infty}^{\infty} p(x)\exp(-\mathrm{i}2\pi s x)\,\mathrm{d}x \\ &= \textstyle\sum_{n=0}^{\infty} \exp(-\alpha)\,\alpha^n \exp(-\mathrm{i}2\pi s n)/n! \\ &= \exp(-\alpha) \textstyle\sum_{n=0}^{\infty} [\alpha \exp(-\mathrm{i}2\pi s)]^n/n! \\ &= \exp[\alpha \exp(-\mathrm{i}2\pi s) - \alpha] \\ &= \exp[\alpha\cos(2\pi s) - \alpha - \mathrm{i}\alpha\sin(2\pi s)]. \qquad (1)\end{aligned}$$

b) As a check on the above result, note that $\psi(s)|_{s=0} = 1$. Next, we compute the derivative of $\psi(s)$ at $s = 0$, as follows:

$$\begin{aligned}\psi'(s)|_{s=0} &= -2\pi\alpha[\sin(2\pi s) + \mathrm{i}\cos(2\pi s)]\exp[\alpha\cos(2\pi s) - \alpha - \mathrm{i}\alpha\sin(2\pi s)]_{s=0} \\ &= -\mathrm{i}2\pi\alpha. \qquad (2)\end{aligned}$$

Comparison with Eq.(29) of Sec.7 shows that $\langle n\rangle = \alpha$. Similarly, the second derivative of $\psi(s)$ at $s = 0$ is found to be

$$\begin{aligned}\psi''(s)|_{s=0} &= \{4\pi^2\alpha^2[\sin(2\pi s) + \mathrm{i}\cos(2\pi s)]^2 - 4\pi^2\alpha[\cos(2\pi s) - \mathrm{i}\sin(2\pi s)]\} \\ &\quad \times \exp[\alpha\cos(2\pi s) - \alpha - \mathrm{i}\alpha\sin(2\pi s)]_{s=0} \\ &= -4\pi^2(\alpha^2 + \alpha). \qquad (3)\end{aligned}$$

Comparison with Eq.(29) of Sec.7 confirms that $\langle n^2\rangle = \alpha^2 + \alpha$, and that, therefore, $\sigma_n^2 = \langle n^2\rangle - \langle n\rangle^2 = \alpha$. The average and the variance of the Poisson distribution thus obtained are seen to be in agreement with the results found directly in Sec.6.

c) The Poisson probability density function consists of a series of equally-spaced delta-functions at unit intervals. It, therefore, resembles a continuous function that has been sampled (at equal intervals) with the aid of a standard comb function. The characteristic function $\psi(s)$ thus becomes a repeated version of the Fourier transform of the underlying continuous function. The period of $\psi(s)$ will be the inverse of the sampling interval, which, in the present case, is 1.0. Thus, we expect $\psi(s)$ to be a periodic function of s, with a period of 1.0. This is indeed the case, as revealed by a quick inspection of Eq.(1).

d) We focus our attention on the behavior of $\psi(s)$ in the vicinity of the point $s = 0$, where each single-period of $\psi(s)$ becomes narrower with an increasing α, as the continuous function underlying Poisson's probability density function becomes wider. Approximating $\sin(2\pi s)$ with $2\pi s$, and $\cos(2\pi s)$ with $1 - 2\pi^2 s^2$ in the vicinity of $s = 0$, we find that, in the neighborhood of $s = 0$, the approximate form of $\psi(s)$ for large values of α is

$$\psi(s) \cong \exp(-2\pi^2\alpha s^2 - \mathrm{i}2\pi\alpha s). \qquad (4)$$

Comparison with Eq.(30) of Sec.7 now shows that the above characteristic function coincides with that of a Gaussian random variable x with average $\langle x\rangle = \alpha$ and variance $\sigma_x^2 = \alpha$. The continuous function underlying Poisson's discrete probability density function is thus seen to approach a Gaussian density function for large values of α.