Solution to Problem 6) a) The discrete binomial probability distribution function is represented by a set of equally spaced delta-functions located at $x = 0, 1, 2, \cdots, N$. Its Fourier transform is evaluated as follows:

$$\begin{aligned}
\psi(s) &= \int_{-\infty}^{\infty} p(x)\exp(-\mathrm{i}2\pi s x)\,\mathrm{d}x \\
&= \sum_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} \exp(-\mathrm{i}2\pi s n) \\
&= (1-p)^N \sum_{n=0}^{N} \binom{N}{n} [p\exp(-\mathrm{i}2\pi s)/(1-p)]^n \\
&= (1-p)^N [1 + p\exp(-\mathrm{i}2\pi s)/(1-p)]^N \\
&= [p\cos(2\pi s) + 1 - p - \mathrm{i}p\sin(2\pi s)]^N. \qquad (1)
\end{aligned}$$

b) As a check on the above result, note that $\psi(s)|_{s=0} = 1$. Next, we compute the derivative of $\psi(s)$ at $s = 0$, as follows:

$$\begin{aligned}
\psi'(s)|_{s=0} &= -2\pi N p[\sin(2\pi s) + \mathrm{i}\cos(2\pi s)][p\cos(2\pi s) + 1 - p - \mathrm{i}p\sin(2\pi s)]^{N-1} \\
&= -\mathrm{i}2\pi N p. \qquad (2)
\end{aligned}$$

Comparison with Eq.(29) shows that $\langle n \rangle = Np$. Similarly, the second derivative of $\psi(s)$ at $s = 0$ is found to be

$$\begin{aligned}
\psi''(s)|_{s=0} &= -4\pi^2 N p[\cos(2\pi s) - \mathrm{i}\sin(2\pi s)][p\cos(2\pi s) - p + 1 - \mathrm{i}p\sin(2\pi s)]^{N-1} \\
&\quad + 4\pi^2 N(N-1)p^2[\sin(2\pi s) + \mathrm{i}\cos(2\pi s)]^2 \\
&\qquad \times [p\cos(2\pi s) + 1 - p - \mathrm{i}p\sin(2\pi s)]^{N-2}|_{s=0} \\
&= -4\pi^2 N p[1 + (N-1)p]. \qquad (3)
\end{aligned}$$

Comparison with Eq.(29) now confirms that $\langle n^2 \rangle = N^2p^2 + Np(1-p)$, and that, therefore, $\sigma_n^2 = \langle n^2 \rangle - \langle n \rangle^2 = Np(1-p)$. The average and the variance of the binomial distribution thus obtained are seen to agree with the results found directly in Sec.6.

c) The binomial probability density function consists of a series of equally-spaced delta-functions at unit intervals. It, therefore, resembles a continuous function that has been sampled (at equal intervals) with the aid of a standard comb function. The characteristic function $\psi(s)$ thus becomes a repeated version of the Fourier transform of the underlying continuous function. The period of $\psi(s)$ will be the inverse of the sampling interval, which, in the present case, is 1.0. Thus, we expect $\psi(s)$ to be a periodic function of s, with a period of 1.0. This is indeed the case, as revealed by a quick inspection of Eq.(1).

d) We focus our attention on the behavior of $\psi(s)$ in the vicinity of the point $s = 0$, where each single-period of $\psi(s)$ becomes narrower with an increasing N, as the continuous function underlying the binomial probability density function becomes wider. Approximating $\sin(2\pi s)$ with $2\pi s$, and $\cos(2\pi s)$ with $1 - 2\pi^2 s^2$ in the vicinity of $s = 0$, we find that, in the neighborhood of $s = 0$, the approximate form of $\psi(s)$ for large values of N is given by

$$\psi(s) \cong (1 - 2p\pi^2 s^2 - \mathrm{i}2\pi p s)^N. \qquad (4)$$

At his point, we attempt to approximate the base expression $(1 - 2p\pi^2 s^2 - \mathrm{i}2\pi ps)$ with e^x, where x is a small entity that depends on p and s. Considering that the first few terms in the Taylor series expansion of e^x yield $\exp(x) \cong 1 + x + ½x^2$, we will have

$$x^2 + 2x + \mathrm{i}4\pi ps(1 - \mathrm{i}\pi s) \cong 0. \quad (5)$$

The only acceptable solution of the quadratic Eq.(5) is $x \cong \sqrt{1 - \mathrm{i}4\pi ps(1 - \mathrm{i}\pi s)} - 1$. Invoking the small ε approximation $\sqrt{1+\varepsilon} \cong 1 + ½\varepsilon - ⅛\varepsilon^2$, and retaining only the terms up to and including 2nd order in s, we now find

$$x \cong \sqrt{1 - \mathrm{i}4\pi ps(1 - \mathrm{i}\pi s)} - 1 \cong -\mathrm{i}2\pi ps(1 - \mathrm{i}\pi s) - ⅛[-\mathrm{i}4\pi ps(1 - \mathrm{i}\pi s)]^2$$

$$\cong -\mathrm{i}2\pi ps(1 - \mathrm{i}\pi s) + 2\pi^2 p^2 s^2 = -2\pi^2 p(1-p)s^2 - \mathrm{i}2\pi ps. \quad (6)$$

The characteristic function of Eq.(4) may now be written as follows:

$$\psi(s) \cong \exp[-2\pi^2 Np(1-p)s^2 - \mathrm{i}2\pi Nps]. \quad (7)$$

Comparison with Eq.(30) of Sec.7 shows that the above characteristic function for the binomial distribution coincides with the corresponding function for a Gaussian random variable x whose average and variance are $\langle x \rangle = Np$ and $\sigma_x^2 = Np(1-p)$, respectively. These, of course, are the average and variance of the binomial distribution. The continuous function underlying the discrete binomial probability density function is thus seen to approach a Gaussian density function for large values of N.