Solution to Problem 5) To prove the necessity of $f''(x)$ being positive everywhere within the domain of a convex cup function, note that

$$f''(x) = \lim_{\Delta x \to 0} [f(x + \Delta x) - 2f(x) + f(x - \Delta x)]/(\Delta x)^2. \tag{1}$$

Let us now set $x_1 = x - \Delta x$ and $x_2 = x + \Delta x$, then choose the parameter λ to be ½. Since $f(x)$ is convex cup, we must have

$$\lambda f(x_1) + (1 - \lambda) f(x_2) \geq f[\lambda x_1 + (1 - \lambda) x_2] \;\rightarrow\; f(x - \Delta x) + f(x + \Delta x) \geq 2f(x). \tag{2}$$

Consequently, the value of $f''(x)$ given by Eq.(1) must be greater than or equal to zero.

To prove that $f''(x) \geq 0$ is a sufficient condition for $f(x)$ to be a convex cup function, we note that $f''(x) \geq 0$ implies that the first derivative $f'(x)$ of $f(x)$ is a non-decreasing function of x. In other words, in any interval $[x_1, x_2]$, the function $f'(x)$ either rises or remains flat as x goes from x_1 to x_2. Picking the arbitrary point x between x_1 and x_2, we may then write

$$\frac{f(x) - f(x_1)}{x - x_1} = \frac{\int_{x_1}^{x} f'(x) \mathrm{d}x}{x - x_1} \leq \frac{\int_{x}^{x_2} f'(x) \mathrm{d}x}{x_2 - x} = \frac{f(x_2) - f(x)}{x_2 - x}. \tag{3}$$

If we now define $\lambda = (x_2 - x)/(x_2 - x_1)$, we will have $(x - x_1)/(x_2 - x_1) = 1 - \lambda$, and, therefore, Eq.(3) can be rewritten as follows:

$$\lambda[f(x) - f(x_1)] \leq (1 - \lambda)[f(x_2) - f(x)]$$

$$\rightarrow \quad \lambda f(x_1) + (1 - \lambda) f(x_2) \geq f(x) = f[\lambda x_1 + (1 - \lambda) x_2]. \tag{4}$$
