Problem 12-27)

a) $F(\vec{k}) = \int_{-\infty}^{\infty} f(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} g(\vec{r}') h(\vec{r}-\vec{r}') d\vec{r}' \right\} e^{-i\vec{k}\cdot\vec{r}} d\vec{r}$

$= \int_{-\infty}^{\infty} g(\vec{r}') \left\{ \int_{-\infty}^{\infty} h(\vec{r}-\vec{r}') e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \right\} d\vec{r}' = \int_{-\infty}^{\infty} g(\vec{r}') \left\{ \int_{-\infty}^{\infty} h(\vec{r}'') e^{-i\vec{k}\cdot(\vec{r}'+\vec{r}'')} d\vec{r}'' \right\} d\vec{r}'$

$= \int_{-\infty}^{\infty} g(\vec{r}') e^{-i\vec{k}\cdot\vec{r}'} \left\{ \int_{-\infty}^{\infty} h(\vec{r}'') e^{-i\vec{k}\cdot\vec{r}''} d\vec{r}'' \right\} d\vec{r}' = G(\vec{k}) H(\vec{k})$ ✓

The proof may be done in reverse as well, as shown below:

$f(\vec{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} F(\vec{k}) e^{+i\vec{k}\cdot\vec{r}} d\vec{k} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} G(\vec{k}) H(\vec{k}) e^{i\vec{k}\cdot\vec{r}} d\vec{k}$

$= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} G(\vec{k}) \left\{ \int_{-\infty}^{\infty} h(\vec{r}') e^{-i\vec{k}\cdot\vec{r}'} d\vec{r}' \right\} e^{i\vec{k}\cdot\vec{r}} d\vec{k} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} h(\vec{r}') \left\{ \int_{-\infty}^{\infty} G(\vec{k}) e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} d\vec{k} \right\} d\vec{r}'$

$= \int_{-\infty}^{\infty} h(\vec{r}') g(\vec{r}-\vec{r}') d\vec{r}' = \int_{-\infty}^{\infty} g(\vec{r}'') h(\vec{r}-\vec{r}'') d\vec{r}''$ ✓

The above convolution theorem can lead to some useful results. For example:

$F(0) = \int_{-\infty}^{\infty} f(\vec{r}) d\vec{r} \implies \int_{-\infty}^{\infty} f(\vec{r}) d\vec{r} = F(0) = G(0) H(0) = \left(\int_{-\infty}^{\infty} g(\vec{r}) d\vec{r} \right) \left(\int_{-\infty}^{\infty} h(\vec{r}) d\vec{r} \right).$

Also note that $g(\vec{r}) * h(\vec{r}) = h(\vec{r}) * g(\vec{r})$.

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b) If $\vec{g}(\vec{r})$ is a vector function of $\vec{r}$, we will have $\vec{F}(\vec{k}) = H(\vec{k}) \vec{G}(\vec{k})$,

where $\vec{G}(\vec{k}) = \int_{-\infty}^{\infty} \vec{g}(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r}$ and $\vec{f}(\vec{r}) = \int_{-\infty}^{\infty} h(\vec{r}') \vec{g}(\vec{r}-\vec{r}') d\vec{r}'$, or,

equivalently, $\vec{f}(\vec{r}) = \int_{-\infty}^{\infty} h(\vec{r}-\vec{r}') \vec{g}(\vec{r}') d\vec{r}'$. These statements can be proven

as before, or by writing $\vec{g}(\vec{r}) = g_x(\vec{r})\hat{x} + g_y(\vec{r})\hat{y} + g_z(\vec{r})\hat{z}$, then applying

the scalar version of the convolution theorem to each component of $\vec{g}(\vec{r})$.

Similarly, if both $\vec{g}(\vec{r})$ and $\vec{h}(\vec{r})$ are vector functions of $\vec{r}$, we will have

1) $f(\vec{r}) = \int_{-\infty}^{\infty} \vec{g}(\vec{r}') \cdot \vec{h}(\vec{r}-\vec{r}') d\vec{r}' \implies F(\vec{k}) = \vec{G}(\vec{k}) \cdot \vec{H}(\vec{k})$

2) $\vec{f}(\vec{r}) = \int_{-\infty}^{\infty} \vec{g}(\vec{r}') \times \vec{h}(\vec{r}-\vec{r}') d\vec{r}' \implies \vec{F}(\vec{k}) = \vec{G}(\vec{k}) \times \vec{H}(\vec{k}).$