

## Problem 12-26)

$$a) f_1(\vec{r}) = \frac{e^{-\alpha r}}{r} \Rightarrow F_1(\vec{k}) = \int_{-\infty}^{\infty} f_1(\vec{r}) e^{-i\vec{k} \cdot \vec{r}} d\vec{r}$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \frac{e^{-\alpha r}}{r} e^{-ikr \cos \theta} 2\pi r^2 \sin \theta dr d\theta$$

$$= 2\pi \int_{r=0}^{\infty} r e^{-\alpha r} \left( \int_{\theta=0}^{\pi} \sin \theta e^{-ikr \cos \theta} d\theta \right) dr$$

$$= 2\pi \int_{r=0}^{\infty} r e^{-\alpha r} \frac{e^{-ikr \cos \theta}}{ikr} \Big|_{\theta=0}^{\pi} dr = \frac{2\pi}{ik} \int_{r=0}^{\infty} [e^{-(\alpha - ik)r} - e^{-(\alpha + ik)r}] dr$$

$$= \frac{2\pi}{ik} \left( \frac{1}{\alpha - ik} - \frac{1}{\alpha + ik} \right) = \frac{2\pi}{ik} \frac{2ik}{\alpha^2 + k^2} \Rightarrow F_1(\vec{k}) = \frac{4\pi}{\alpha^2 + k^2}$$

← In general  $\alpha$  is complex, with a positive real part.

In the limit when  $\alpha \rightarrow 0$ , we have  $f_1(\vec{r}) = 1/r \Leftrightarrow F_1(\vec{k}) = \frac{4\pi}{k^2}$ .

$$b) \vec{f}_2(\vec{r}) = \frac{e^{-\alpha r}}{r} \hat{r} \Rightarrow \vec{F}_2(\vec{k}) = \int_{-\infty}^{\infty} \frac{e^{-\alpha r}}{r} \hat{r} e^{-i\vec{k} \cdot \vec{r}} d\vec{r}$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \frac{e^{-\alpha r}}{r} e^{-ikr \cos \theta} (\cos \theta \hat{k}) 2\pi r^2 \sin \theta dr d\theta = 2\pi \hat{k} \int_{r=0}^{\infty} r e^{-\alpha r} \left( \int_{\theta=0}^{\pi} \sin \theta \cos \theta e^{-ikr \cos \theta} d\theta \right) dr$$

$$= 2\pi \hat{k} \int_{r=0}^{\infty} r e^{-\alpha r} \left\{ \frac{\cos \theta e^{-ikr \cos \theta}}{ikr} \Big|_{\theta=0}^{\pi} + \int_{\theta=0}^{\pi} \frac{\sin \theta e^{-ikr \cos \theta}}{ikr} d\theta \right\} dr$$

← use integration by parts

$$= \frac{2\pi \hat{k}}{ik} \int_{r=0}^{\infty} e^{-\alpha r} \left\{ -e^{ikr} - e^{-ikr} + \frac{e^{ikr} - e^{-ikr}}{ikr} \right\} dr = \frac{2\pi \hat{k}}{ik} \left\{ \frac{-1}{\alpha - ik} - \frac{1}{\alpha + ik} + \int_{r=0}^{\infty} \frac{2e^{-\alpha r} \sin(ikr)}{kr} dr \right\}$$

$$= -\frac{4\pi \alpha \hat{k}}{ik(k^2 + \alpha^2)} + \frac{4\pi \hat{k}}{ik^2} \int_{r=0}^{\infty} e^{-\alpha r} \frac{\sin(ikr)}{r} dr = -\frac{4\pi \alpha \hat{k}}{ik(k^2 + \alpha^2)} + \frac{4\pi \hat{k}}{ik^2} \arctan\left(\frac{k}{\alpha}\right)$$

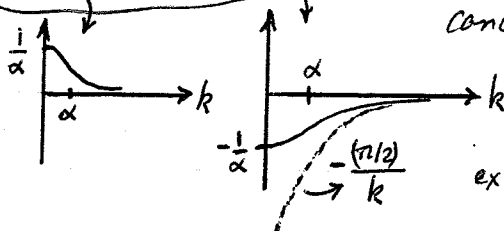
$$\Rightarrow \vec{F}_2(\vec{k}) = i \frac{4\pi \hat{k}}{k} \left( \frac{\alpha}{k^2 + \alpha^2} - \frac{\tan^{-1}(k/\alpha)}{k} \right)$$

Case of  $\alpha \rightarrow 0$ :

In the vicinity of  $k=0$  the two functions cancel each other out. When  $k \gg \alpha$ , the second function dominates. Thus

when  $\alpha \rightarrow 0$  we have  $\vec{F}_2(\vec{k}) \rightarrow -i \frac{2\pi \hat{k}}{k^2}$  except around  $k=0$  where  $\vec{F}_2(\vec{k}) \rightarrow 0$ .

For small  $\alpha > 0$ :



$$\begin{aligned}
 c) \quad f_3(\vec{r}) &= \frac{1}{r^2} \Rightarrow \vec{F}_3(\vec{k}) = \int_{-\infty}^{\infty} \frac{1}{r^2} e^{-i\vec{k} \cdot \vec{r}} d\vec{r} = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \frac{1}{r^2} e^{-ikr \cos \theta} 2\pi r^2 \sin \theta dr d\theta \\
 &= 2\pi \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \sin \theta e^{-ikr \cos \theta} d\theta dr = 2\pi \int_{r=0}^{\infty} \frac{\exp(-ikr \cos \theta) \Big|_{\theta=0}^{\pi}}{ikr} dr \\
 &= 2\pi \int_0^{\infty} \frac{e^{ikr} - e^{-ikr}}{ikr} dr = 4\pi \int_0^{\infty} \frac{\sin(kr)}{kr} dr = \frac{4\pi}{k} \int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{2\pi^2}{k}
 \end{aligned}$$

$$\begin{aligned}
 d) \quad f_4(\vec{r}) &= -\hat{r}/r^2 \Rightarrow \vec{F}_4(\vec{k}) = \int_{-\infty}^{\infty} \left(\frac{-\hat{r}}{r^2}\right) e^{-i\vec{k} \cdot \vec{r}} d\vec{r} = - \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \frac{e^{-ikr \cos \theta}}{r^2} (\cos \theta \hat{k}) 2\pi r^2 \sin \theta dr d\theta \\
 &= -2\pi \hat{k} \int_{r=0}^{\infty} dr \int_{\theta=0}^{\pi} \sin \theta \cos \theta e^{-ikr \cos \theta} d\theta = -2\pi \hat{k} \int_{r=0}^{\infty} dr \int_{x=-1}^{+1} x e^{-ikrx} dx \\
 &= -2\pi \hat{k} \int_{r=0}^{\infty} dr \left\{ \frac{x e^{-ikrx}}{-ikr} \Big|_{x=-1}^{+1} - \int_{x=-1}^{+1} \frac{e^{-ikrx}}{-ikr} dx \right\} \quad \leftarrow \text{use integration by parts} \\
 &= -2\pi \hat{k} \int_0^{\infty} dr \left\{ \frac{2 \cos(kr)}{-ikr} - \frac{2i \sin(kr)}{k^2 r^2} \right\} = \frac{4\pi i \hat{k}}{k} \int_0^{\infty} \frac{\sin(x) - x \cos(x)}{x^2} dx = -\frac{4\pi i \hat{k}}{k} \left( \frac{\sin x}{x} \right) \Big|_0^{\infty} \\
 &\Rightarrow \vec{F}_4(\vec{k}) = \frac{4\pi i \hat{k}}{k}
 \end{aligned}$$

$$\text{Digression: } \mathcal{F}\left\{\frac{1}{r}\right\} = \frac{4\pi}{k^2} \Rightarrow \frac{1}{r} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{4\pi}{k^2} e^{+i\vec{k} \cdot \vec{r}} d\vec{k} \Rightarrow$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2} = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{i\vec{k}}{k^2} e^{i\vec{k} \cdot \vec{r}} d\vec{k} \Rightarrow \vec{\nabla} \cdot \vec{\nabla}\left(\frac{1}{r}\right) = -\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{i^2 \vec{k} \cdot \vec{k}}{k^2} e^{i\vec{k} \cdot \vec{r}} d\vec{k}$$

$$\Rightarrow \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = \frac{4\pi}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\vec{k} \cdot \vec{r}} d\vec{k} = 4\pi \delta(\vec{r})$$