Problem 12-25)

a) $\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{r},t) = \vec{\nabla}\times\vec{\nabla}\times\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\vec{A}(\vec{k},\omega)\,e^{+i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{k}\,d\omega$

$$= \frac{i^2}{(2\pi)^4}\int_{-\infty}^{\infty}\vec{k}\times[\vec{k}\times\vec{A}(\vec{k},\omega)]\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{k}\,d\omega$$

We now use the vector identity $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$ to find:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{r},t) = -\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}[\vec{k}\cdot\vec{A}(\vec{k},\omega)]\vec{k}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{k}\,d\omega + \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}k^2\vec{A}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{k}\,d\omega$$

✓ The first term on the right-hand side is the gradient of the divergence of $\vec{A}(\vec{r},t)$. As soon as we take the divergence, we get $\vec{k}\cdot\vec{A}(\vec{k},\omega)$ under the integral. This operation retains the longitudinal component $\vec{A}_{\parallel}(\vec{k},\omega)$, but throws away the perpendicular component $\vec{A}_{\perp}(\vec{k},\omega)$. We thus have:

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = -\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}[\vec{k}\cdot\vec{A}(\vec{k},\omega)]\vec{k}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{k}\,d\omega = -\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}k^2\vec{A}_{\parallel}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{k}\,d\omega.$$

✓ The second term is substantively different than the first. In the second term both $\vec{A}_{\parallel}$ and $\vec{A}_{\perp}$ are retained under the integral sign; however, the entire $\vec{A}(\vec{k},\omega)$ is multiplied by k^2. The second term is thus defined to be the Laplacian of $\vec{A}(\vec{r},t)$:

$$\vec{\nabla}^2\vec{A}(\vec{r},t) = -\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}k^2\vec{A}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{k}\,d\omega.$$

We thus have: $\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{r},t) = \vec{\nabla}[\vec{\nabla}\cdot\vec{A}(\vec{r},t)] - \vec{\nabla}^2\vec{A}(\vec{r},t)$. Note that this is not so much a mathematical theorem as it is a definition of the Laplacian operator.

While the Laplacian of a scalar function, say, $\vec{\nabla}^2\psi(\vec{r},t)$, is readily understood to be the divergence of the gradient of ψ (because this is the only way in which the $\vec{\nabla}$ operator can be applied twice to the scalar function), the Laplacian of a vector function, say, $\vec{\nabla}^2\vec{A}(\vec{r},t)$ is not what immediately comes to mind.

To apply the $\vec{\nabla}$ operator twice to $\vec{A}(\vec{r},t)$, one has a couple of choices. One is to calculate $\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$, but this was already discussed above, and is not what has traditionally been called the Laplacian. Another option is $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})$, but this is identically zero. [One way to see this is to note that $\vec{k}\cdot[\vec{k}\times\vec{A}(\vec{k},\omega)]=0$, because $\vec{k}\times\vec{A}$ is $\perp$ to $\vec{k}$.]. A third option would be to write the Laplacian as $\vec{\nabla}\times\vec{\nabla}\times\vec{A}$, but this option was not chosen either. Instead, Laplacian was *defined* as a combination of the two viable options, namely, $\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}\times\vec{\nabla}\times\vec{A}$. In the Fourier domain, this definition leads to a simple multiplication of $-k^2$ into $\vec{A}(\vec{k},\omega)$, as discussed above.

In Cartesian coordinates we can write:

$$\vec{\nabla}^2\vec{A}(\vec{r},t)=-\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}k^2\vec{A}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{k}\,d\omega =$$

$$=-\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}k^2\left[A_x(\vec{k},\omega)\hat{x}+A_y(\vec{k},\omega)\hat{y}+A_z(\vec{k},\omega)\hat{z}\right]e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{k}\,d\omega$$

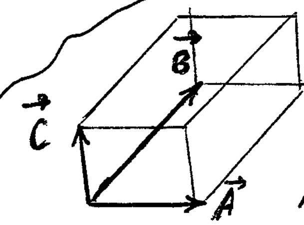
$$=\left[\vec{\nabla}^2A_x(\vec{r},t)\right]\hat{x}+\left[\vec{\nabla}^2A_y(\vec{r},t)\right]\hat{y}+\left[\vec{\nabla}^2A_z(\vec{r},t)\right]\hat{z} \quad \checkmark$$

This is the usual definition of $\vec{\nabla}^2\vec{A}$ in terms of the Laplacians of the three scalar functions A_x, A_y, and A_z.

But in coordinate systems other than Cartesian, the correct way to go about calculating $\vec{\nabla}^2\vec{A}(\vec{r},t)$ is to use the definition $\vec{\nabla}^2\vec{A}=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}\times\vec{\nabla}\times\vec{A}$. Do not think that, for instance, in cylindrical coordinates you can write $\vec{\nabla}^2\vec{A}(\vec{r},t)=(\vec{\nabla}^2A_\rho)\hat{\rho}+(\vec{\nabla}^2A_\phi)\hat{\phi}+(\vec{\nabla}^2A_z)\hat{z}$. This is manifestly wrong! The (ρ,ϕ,z) components of a vector field in the $(\vec{r},t)$ domain are not in a one-to-one relation with the (ρ,ϕ,z) components of the Fourier transform of the field in the $(\vec{k},\omega)$ domain.

b) $\vec{B}(\vec{r})\cdot[\vec{\nabla}\times\vec{A}(\vec{r})] = \frac{1}{(2\pi)^6}\int_{-\infty}^{\infty}\vec{B}(\vec{k})e^{i\vec{k}\cdot\vec{r}}d\vec{k}\cdot\int_{-\infty}^{\infty}i\vec{k}'\times\vec{A}(\vec{k}')e^{i\vec{k}'\cdot\vec{r}}d\vec{k}'$

$$= \frac{i}{(2\pi)^6}\iint_{-\infty}^{\infty}\vec{B}(\vec{k})\cdot[\vec{k}'\times\vec{A}(\vec{k}')]e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}d\vec{k}\,d\vec{k}'$$



Using the vector identity $(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{B}\times\vec{C})\cdot\vec{A}$ ← (Volume of a parallel-piped with sides $\vec{A}$, $\vec{B}$ and $\vec{C}$)

We may write:

$$\vec{B}\cdot(\vec{\nabla}\times\vec{A}) = \frac{i}{(2\pi)^6}\iint_{-\infty}^{\infty}\vec{k}'\cdot[\vec{A}(\vec{k}')\times\vec{B}(k)]e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}d\vec{k}\,d\vec{k}'.$$

Similarly: $\vec{A}(\vec{r})\cdot[\vec{\nabla}\times\vec{B}(\vec{r})] = \frac{1}{(2\pi)^6}\int_{-\infty}^{\infty}\vec{A}(\vec{k}')e^{i\vec{k}'\cdot\vec{r}}d\vec{k}'\cdot\int_{-\infty}^{\infty}i\vec{k}\times\vec{B}(\vec{k})e^{i\vec{k}\cdot\vec{r}}d\vec{k}$

$$= \frac{i}{(2\pi)^6}\iint_{-\infty}^{\infty}\vec{k}\cdot[\vec{B}(\vec{k})\times\vec{A}(\vec{k}')]e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}d\vec{k}\,d\vec{k}'$$

$$\Rightarrow \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) = \frac{i}{(2\pi)^6}\iint_{-\infty}^{\infty}(\vec{k}+\vec{k}')[A(\vec{k}')\times B(\vec{k})]e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}d\vec{k}\,d\vec{k}'$$

$$= \vec{\nabla}\cdot\left\{\frac{1}{(2\pi)^6}\iint_{-\infty}^{\infty}\vec{A}(\vec{k}')\times\vec{B}(\vec{k})\,e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}d\vec{k}\,d\vec{k}'\right\}$$

$$= \vec{\nabla}\cdot\left\{\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\vec{A}(\vec{k}')e^{i\vec{k}'\cdot\vec{r}}d\vec{k}' \times \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\vec{B}(\vec{k})e^{i\vec{k}\cdot\vec{r}}d\vec{k}\right\} = \vec{\nabla}\cdot[\vec{A}(\vec{r})\times\vec{B}(\vec{r})]$$