Problem 12-21) Green's first identity was derived in Problem 20 by starting with the vector-field $\phi\boldsymbol{\nabla}\psi$. The same procedure may be followed for the vector-field $\psi\,\boldsymbol{\nabla}\phi$. The two identities thus obtained are given below:

$$\int_{\text{volume}} \{\phi(\boldsymbol{r})\boldsymbol{\nabla}^2\psi(\boldsymbol{r}) + [\boldsymbol{\nabla}\phi(\boldsymbol{r})]\cdot[\boldsymbol{\nabla}\psi(\boldsymbol{r})]\}\,\mathrm{d}v = \oint_{\text{surface}} \phi(\boldsymbol{r})\frac{\partial\psi(\boldsymbol{r})}{\partial\boldsymbol{n}}\,\mathrm{d}s,$$

$$\int_{\text{volume}} \{\psi(\boldsymbol{r})\boldsymbol{\nabla}^2\phi(\boldsymbol{r}) + [\boldsymbol{\nabla}\psi(\boldsymbol{r})]\cdot[\boldsymbol{\nabla}\phi(\boldsymbol{r})]\}\,\mathrm{d}v = \oint_{\text{surface}} \psi(\boldsymbol{r})\frac{\partial\phi(\boldsymbol{r})}{\partial\boldsymbol{n}}\,\mathrm{d}s.$$

Subtracting the second of the above identities from the first, we arrive at Green's theorem, namely,

$$\int_{\text{volume}} [\phi(\boldsymbol{r})\boldsymbol{\nabla}^2\psi(\boldsymbol{r}) - \psi(\boldsymbol{r})\boldsymbol{\nabla}^2\phi(\boldsymbol{r})]\,\mathrm{d}v = \oint_{\text{surface}} [\phi(\boldsymbol{r})\frac{\partial\psi(\boldsymbol{r})}{\partial\boldsymbol{n}} - \psi(\boldsymbol{r})\frac{\partial\phi(\boldsymbol{r})}{\partial\boldsymbol{n}}]\,\mathrm{d}s.$$