Problem 12-20) Applying Gauss's theorem to the vector-field $\phi\boldsymbol{\nabla}\psi$ yields:

$$\int_{\text{volume}} \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\psi)\mathrm{d}v = \int_{\text{surface}}(\phi\boldsymbol{\nabla}\psi)\cdot\mathrm{d}\boldsymbol{s} = \oint_{\text{surface}}\phi(\boldsymbol{r})\frac{\partial\psi(\boldsymbol{r})}{\partial\boldsymbol{n}}\mathrm{d}s.$$

Next, we apply the identity $\boldsymbol{\nabla}\cdot(\psi\boldsymbol{A}) = (\boldsymbol{\nabla}\psi)\cdot\boldsymbol{A} + \psi\,\boldsymbol{\nabla}\cdot\boldsymbol{A}$, proved in Problem 14(a), to the left-hand-side of the above equation. We will have

$$\int_{\text{volume}} \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\psi)\mathrm{d}v = \int_{\text{volume}}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi + \phi\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi)\mathrm{d}v = \int_{\text{volume}}\{\phi(\boldsymbol{r})\boldsymbol{\nabla}^2\psi(\boldsymbol{r}) + [\boldsymbol{\nabla}\phi(\boldsymbol{r})]\cdot[\boldsymbol{\nabla}\psi(\boldsymbol{r})]\}\mathrm{d}v.$$

This completes the proof of Green's first identity.