Problem 15

a) $\vec{\nabla}\times(\vec{\nabla}\psi) = \vec{\nabla}\times\left(\frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial y}\hat{y} + \frac{\partial\psi}{\partial z}\hat{z}\right)$

$$= \left(\frac{\partial^2\psi}{\partial y\partial z} - \frac{\partial^2\psi}{\partial z\partial y}\right)\hat{x} + \left(\frac{\partial^2\psi}{\partial z\partial x} - \frac{\partial^2\psi}{\partial x\partial z}\right)\hat{y} + \left(\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x}\right)\hat{z} = 0 \checkmark$$

b) $\vec{\nabla}\times(\psi\vec{A}) = \left[\frac{\partial}{\partial y}(\psi A_z) - \frac{\partial}{\partial z}(\psi A_y)\right]\hat{x} + \left[\frac{\partial}{\partial z}(\psi A_x) - \frac{\partial}{\partial x}(\psi A_z)\right]\hat{y}$

$$+ \left[\frac{\partial}{\partial x}(\psi A_y) - \frac{\partial}{\partial y}(\psi A_x)\right]\hat{z} = \left(A_z\frac{\partial\psi}{\partial y} + \psi\frac{\partial A_z}{\partial y} - A_y\frac{\partial\psi}{\partial z} - \psi\frac{\partial A_y}{\partial z}\right)\hat{x}$$

$$+ \left(A_x\frac{\partial\psi}{\partial z} + \psi\frac{\partial A_x}{\partial z} - A_z\frac{\partial\psi}{\partial x} - \psi\frac{\partial A_z}{\partial x}\right)\hat{y} + \left(A_y\frac{\partial\psi}{\partial x} + \psi\frac{\partial A_y}{\partial x} - A_x\frac{\partial\psi}{\partial y} - \psi\frac{\partial A_x}{\partial y}\right)\hat{z}$$

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$$= \left(A_z \frac{\partial \psi}{\partial y} - A_y \frac{\partial \psi}{\partial z}\right)\hat{x} + \left(A_x \frac{\partial \psi}{\partial z} - A_z \frac{\partial \psi}{\partial x}\right)\hat{y} + \left(A_y \frac{\partial \psi}{\partial x} - A_x \frac{\partial \psi}{\partial y}\right)\hat{z}$$

$$+ \psi\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}\right]$$

$$= (\vec{\nabla}\psi) \times \vec{A} + \psi(\vec{\nabla} \times \vec{A}) \quad ✓$$

c) $$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}\right]$$

$$= \left(\frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x \partial z}\right)\hat{x} + \left(\frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial y \partial x}\right)\hat{y}$$

$$+ \left(\frac{\partial^2 A_x}{\partial z \partial x} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z}\right)\hat{z}$$

To the first term of the above expression we add $\pm \frac{\partial^2 A_x}{\partial x^2}$, to the second term $\pm \frac{\partial^2 A_y}{\partial y^2}$, to the third term $\pm \frac{\partial^2 A_z}{\partial z^2}$. We'll have:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \left[\frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{\partial A_x}{\partial x}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)A_x\right]\hat{x}$$

$$+ \left[\frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial z} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)A_y\right]\hat{y}$$

$$+ \left[\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)A_z\right]\hat{z}$$

$$= \frac{\partial}{\partial x}(\vec{\nabla} \cdot \vec{A})\hat{x} + \frac{\partial}{\partial y}(\vec{\nabla} \cdot \vec{A})\hat{y} + \frac{\partial}{\partial z}(\vec{\nabla} \cdot \vec{A})\hat{z} - (\vec{\nabla}^2 A_x)\hat{x} - (\vec{\nabla}^2 A_y)\hat{y} - (\vec{\nabla}^2 A_z)\hat{z}$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} \quad ✓$$

d) $$\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{\nabla} \times [(A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} +$$

$$(A_x B_y - A_y B_x)\hat{z}] = \left[\frac{\partial}{\partial y}(A_x B_y - A_y B_x) - \frac{\partial}{\partial z}(A_z B_x - A_x B_z)\right]\hat{x}$$

$$+ \left[\frac{\partial}{\partial z}(A_y B_z - A_z B_y) - \frac{\partial}{\partial x}(A_x B_y - A_y B_x)\right]\hat{y} + \left[\frac{\partial}{\partial x}(A_z B_x - A_x B_z)\right.$$

$$\left. - \frac{\partial}{\partial y}(A_y B_z - A_z B_y)\right]\hat{z} = \left[\cancel{B_y \frac{\partial A_x}{\partial y}} + \cancel{A_x \frac{\partial B_y}{\partial y}} - \cancel{B_x \frac{\partial A_y}{\partial y}} - \cancel{A_y \frac{\partial B_x}{\partial y}}\right.$$

$$-\cancel{B_x\frac{\partial A_z}{\partial z}} - \cancel{A_z\frac{\partial B_x}{\partial z}} + \cancel{B_z\frac{\partial A_x}{\partial z}} + \cancel{A_x\frac{\partial B_z}{\partial z}}\Big]\hat{x} + \Big[\cancel{B_z\frac{\partial A_y}{\partial z}} + \cancel{A_y\frac{\partial B_z}{\partial z}}$$

$$-\cancel{B_y\frac{\partial A_z}{\partial z}} - A_z\frac{\partial B_y}{\partial z} - \cancel{B_y\frac{\partial A_x}{\partial x}} - \cancel{A_x\frac{\partial B_y}{\partial x}} + \cancel{B_x\frac{\partial A_y}{\partial x}} + \cancel{A_y\frac{\partial B_x}{\partial x}}\Big]\hat{y}$$

$$+\Big[\cancel{B_x\frac{\partial A_z}{\partial x}} + \cancel{A_z\frac{\partial B_x}{\partial x}} - \cancel{B_z\frac{\partial A_x}{\partial x}} - \cancel{A_x\frac{\partial B_z}{\partial x}} - \cancel{B_z\frac{\partial A_y}{\partial y}} - \cancel{A_y\frac{\partial B_z}{\partial y}} + \cancel{B_y\frac{\partial A_z}{\partial y}}$$

$$+\cancel{A_z\frac{\partial B_y}{\partial y}}\Big]\hat{z} =$$

add and subtract

$$\Big[A_x\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) - A_x\frac{\partial B_x}{\partial x} - B_x\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + B_x\frac{\partial A_x}{\partial x}$$

$$+B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z} - A_y\frac{\partial B_x}{\partial y} - A_z\frac{\partial B_x}{\partial z}\Big]\hat{x} + \Big[A_y\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)$$

$$-A_y\frac{\partial B_y}{\partial y} - B_y\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + B_y\frac{\partial A_y}{\partial y} + B_x\frac{\partial A_y}{\partial x} + B_z\frac{\partial A_y}{\partial z} - A_x\frac{\partial B_y}{\partial x}$$

$$-A_z\frac{\partial B_y}{\partial z}\Big]\hat{y} + \Big[A_z\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) - A_z\frac{\partial B_z}{\partial z} - B_z\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$+B_z\frac{\partial A_z}{\partial z} + B_x\frac{\partial A_z}{\partial x} + B_y\frac{\partial A_z}{\partial y} - A_x\frac{\partial B_z}{\partial x} - A_y\frac{\partial B_z}{\partial y}\Big]\hat{z} =$$

$$\Big[A_x(\vec{\nabla}\cdot\vec{B}) - B_x(\vec{\nabla}\cdot\vec{A}) + \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)A_x - \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_x\Big]\hat{x}$$

$$+\Big[A_y(\vec{\nabla}\cdot\vec{B}) - B_y(\vec{\nabla}\cdot\vec{A}) + \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)A_y - \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_y\Big]\hat{y}$$

$$+\Big[A_z(\vec{\nabla}\cdot\vec{B}) - B_z(\vec{\nabla}\cdot\vec{A}) + \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)A_z - \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_z\Big]\hat{z}$$

$$= (\vec{\nabla}\cdot\vec{B})(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) - (\vec{\nabla}\cdot\vec{A})(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) + (\vec{B}\cdot\vec{\nabla})(A_x\hat{x} + A_y\hat{y} + A_z\hat{z})$$

$$-(\vec{A}\cdot\vec{\nabla})(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) \Longrightarrow$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{\nabla}\cdot\vec{B})\vec{A} - (\vec{\nabla}\cdot\vec{A})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} \quad ✓$$