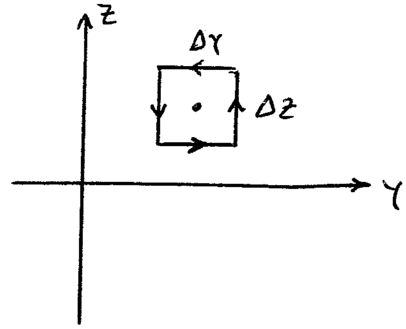
Problem 12-11) a) Cartesian Coordinates:

$$\vec{\nabla}\times\vec{f}(x,y,z) = \frac{\oint \vec{f}\cdot d\vec{\ell}}{\Delta s \to 0} \quad (3\ \text{Components})$$



x- component of Curl $\vec{f} = \frac{1}{\Delta y \Delta z}\{ f_z(x, y+\frac{1}{2}\Delta y, z)\Delta z$

$- f_y(x,y,z+\frac{1}{2}\Delta z)\Delta y - f_z(x, y-\frac{1}{2}\Delta y, z)\Delta z + f_y(x,y,z-\frac{1}{2}\Delta z)\Delta y\} = \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}.$

The y- and z-Components of $\vec{\nabla}\times\vec{f}$ may be obtained in a similar way.

Consequently: $\vec{\nabla}\times\vec{f}(\vec{r}) = (\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z})\hat{x} + (\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x})\hat{y} + (\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y})\hat{z}.$

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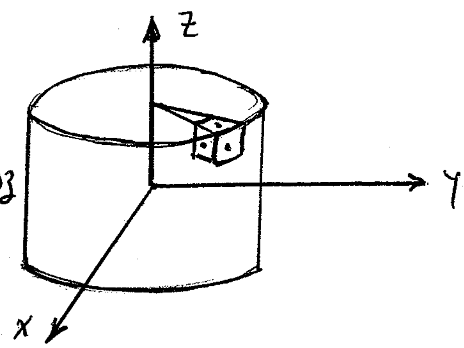
b) Cylindrical Coordinates:



ρ- Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\rho\Delta\phi\Delta z}\{ f_z(\rho, \phi+\frac{\Delta\phi}{2}, z)\Delta z$

$- f_\phi(\rho,\phi,z+\frac{\Delta z}{2})\rho\Delta\phi - f_z(\rho,\phi-\frac{\Delta\phi}{2},z)\Delta z$

$+ f_\phi(\rho,\phi,z-\frac{\Delta z}{2})\rho\Delta\phi\} = \frac{1}{\rho}\frac{\partial}{\partial\phi} f_z - \frac{\partial f_\phi}{\partial z}.$

ϕ- Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\Delta\rho\Delta z}\{ -f_z(\rho+\frac{\Delta\rho}{2}, \phi, z)\Delta z + f_\rho(\rho,\phi,z+\frac{\Delta z}{2})\Delta\rho$

$+ f_z(\rho - \frac{\Delta\rho}{2}, \phi, z)\Delta z - f_\rho(\rho,\phi,z-\frac{\Delta z}{2})\} = \frac{-\partial}{\partial\rho} f_z + \frac{\partial}{\partial z} f_\rho.$

z- Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\rho\Delta\rho\Delta\phi}\{ f_\phi(\rho+\frac{\Delta\rho}{2}, \phi, z)(\rho+\frac{\Delta\rho}{2})\Delta\phi - f_\rho(\rho,\phi+\frac{\Delta\phi}{2}, z)\Delta\rho$

$- f_\phi(\rho - \frac{\Delta\rho}{2}, \phi, z)(\rho - \frac{\Delta\rho}{2})\Delta\phi + f_\rho(\rho, \phi+\frac{\Delta\phi}{2}, z)\Delta\rho\} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\phi) - \frac{1}{\rho}\frac{\partial}{\partial\phi} f_\rho.$

Therefore, $\vec{\nabla}\times\vec{f}(\vec{r}) = (\frac{1}{\rho}\frac{\partial}{\partial\phi} f_z - \frac{\partial}{\partial z} f_\phi)\hat{\rho} + (\frac{\partial}{\partial z} f_\rho - \frac{\partial}{\partial\rho} f_z)\hat{\phi} + \frac{1}{\rho}(\frac{\partial}{\partial\rho}(\rho f_\phi) - \frac{\partial}{\partial\phi} f_\rho)\hat{z}.$

c) Spherical Coordinates:

ρ- Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\rho^2 \sin\theta\,\Delta\theta\,\Delta\phi}\Big\{-f_\theta(\rho,\theta,\phi+\frac{\Delta\phi}{2})\rho\Delta\theta -$

$f_\phi(\rho,\theta-\frac{\Delta\theta}{2},\phi)\rho\sin(\theta-\frac{\Delta\theta}{2})\Delta\phi + f_\theta(\rho,\theta,\phi-\frac{\Delta\phi}{2})\rho\Delta\theta + f_\phi(\rho,\theta+\frac{\Delta\theta}{2},\phi)\rho\sin(\theta+\frac{\Delta\theta}{2})\Delta\phi\Big\}$

$= \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\phi) - \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}f_\theta.$

θ- Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\rho\sin\theta\,\Delta\phi\,\Delta\rho}\Big\{-f_\phi(\rho+\frac{\Delta\rho}{2},\theta,\phi)(\rho+\frac{\Delta\rho}{2})\sin\theta\,\Delta\phi$

$-f_\rho(\rho,\theta,\phi-\frac{\Delta\phi}{2})\Delta\rho + f_\phi(\rho-\frac{\Delta\rho}{2},\theta,\phi)(\rho-\frac{\Delta\rho}{2})\sin\theta\,\Delta\phi + f_\rho(\rho,\theta,\phi+\frac{\Delta\phi}{2})\Delta\rho\Big\}$

$= \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}f_\rho - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\phi).$

ϕ- Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\rho\,\Delta\theta\,\Delta\rho}\Big\{f_\theta(\rho+\frac{\Delta\rho}{2},\theta,\phi)(\rho+\frac{\Delta\rho}{2})\Delta\theta$

$-f_\rho(\rho,\theta+\frac{\Delta\theta}{2},\phi)\Delta\rho - f_\theta(\rho-\frac{\Delta\rho}{2},\theta,\phi)(\rho-\frac{\Delta\rho}{2})\Delta\theta + f_\rho(\rho,\theta-\frac{\Delta\theta}{2},\phi)\Delta\rho\Big\}$

$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\theta) - \frac{1}{\rho}\frac{\partial}{\partial\theta}f_\rho.$

Therefore, $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\rho\sin\theta}\Big[\frac{\partial}{\partial\theta}(\sin\theta f_\phi) - \frac{\partial}{\partial\phi}f_\theta\Big]\hat{\rho} + \frac{1}{\rho}\Big[\frac{1}{\sin\theta}\frac{\partial f_\rho}{\partial\phi} - \frac{\partial}{\partial\rho}(\rho f_\phi)\Big]\hat{\theta}$

$+ \frac{1}{\rho}\Big[\frac{\partial}{\partial\rho}(\rho f_\theta) - \frac{\partial}{\partial\theta}f_\rho\Big]\hat{\phi}.$