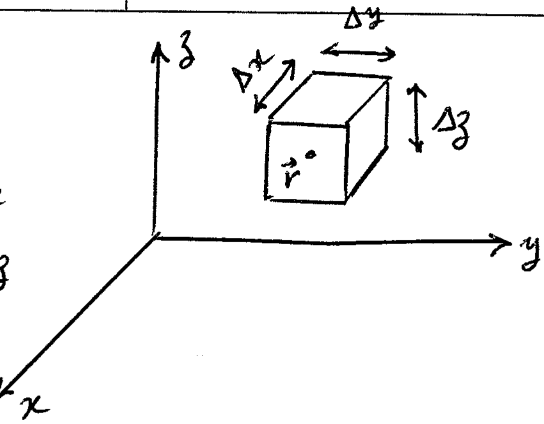
Problem 12-10) a) $\vec{\nabla}\cdot\vec{f}(\vec{r}) = \dfrac{\oint \vec{f}(\vec{r})\cdot d\vec{s}}{\Delta V_{\to 0}}$.



Surface integral on the right-hand-side surface of the cube $= f_y(x, y+\frac{\Delta y}{2}, z)\,\Delta x\Delta z$

Surface integral on the left-hand-side surface of the cube $= -f_y(x, y-\frac{1}{2}\Delta y, z)\,\Delta x\,\Delta z$

Add the above contributions to the surface integral, then divide by $\Delta V=\Delta x\,\Delta y\,\Delta z$, and you'll obtain $\frac{\partial}{\partial y} f_y(x,y,z)$. Similarly, for the front and rear surfaces you'll find $\frac{\partial}{\partial x} f_x(x,y,z)$, and for the top and bottom surfaces you'll get $\frac{\partial}{\partial z} f_z(x,y,z)$. When you add them all up, you'll have $\vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z$.

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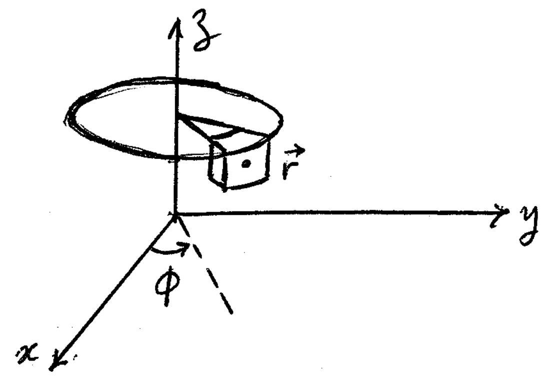
b) Cylindrical Coordinates: $\vec{f}(\rho, \phi, z)$

$\vec{\nabla}\cdot\vec{f}(\vec{r}) = \dfrac{\oint \vec{f}(\vec{r})\cdot d\vec{s}}{\Delta V_{\to 0}}$.



Contribution of front surface to the integral $= f_\rho(\rho+\frac{1}{2}\Delta\rho, \phi, z)(\rho+\frac{1}{2}\Delta\rho)\Delta\phi\,\Delta z$

Contribution of rear surface to the integral $= -f_\rho(\rho-\frac{1}{2}\Delta\rho, \phi, z)(\rho-\frac{1}{2}\Delta\rho)\Delta\phi\,\Delta z$

Add the above contributions to the total surface integral, then divide by $\Delta V = \rho\Delta\phi\Delta\rho\Delta z$, and you'll obtain $\frac{1}{\rho}\frac{\partial}{\partial\rho}[\rho f_\rho(\rho,\phi,z)] = \frac{1}{\rho}f_\rho + \frac{\partial}{\partial\rho}f_\rho$

Similarly, the contributions of the right and left surfaces will yield

$$\frac{1}{\rho\Delta\phi\Delta\rho\Delta z}\left\{f_\phi(\rho, \phi+\tfrac{\Delta\phi}{2}, z)\Delta\rho\Delta z - f_\phi(\rho, \phi-\tfrac{\Delta\phi}{2}, z)\Delta\rho\Delta z\right\} = \frac{1}{\rho}\frac{\partial}{\partial\phi}f_\phi$$

As for the top and bottom surfaces, we'll have

$$\frac{1}{\rho\Delta\phi\Delta\rho\Delta z}\left\{f_z(\rho,\phi,z+\tfrac{\Delta z}{2})\,\rho\Delta\phi\Delta\rho - f_z(\rho,\phi,z-\tfrac{\Delta z}{2})\,\rho\Delta\phi\Delta\rho\right\} = \frac{\partial}{\partial z} f_z$$

Consequently: $$\vec{\nabla}\cdot\vec{f}(\vec{r}) = \frac{1}{\rho} f_\rho(\vec{r}) + \frac{\partial}{\partial \rho} f_\rho(\vec{r}) + \frac{1}{\rho}\frac{\partial}{\partial\phi} f_\phi(\vec{r}) + \frac{\partial}{\partial z} f_z(\vec{r}).$$

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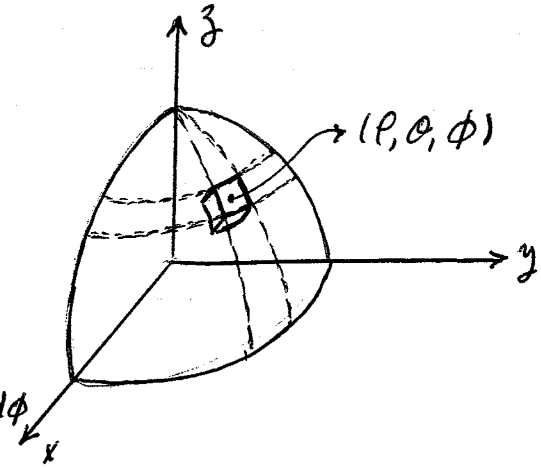
c) Spherical Coordinates: $f(\rho,\theta,\phi)$

$$\vec{\nabla}\cdot\vec{f}(\vec{r}) = \frac{\oint \vec{f}(\vec{r})\cdot d\vec{s}}{\underset{\to 0}{\Delta V}}$$



Contribution of front surface to the integral $= f_\rho(\rho+\frac{1}{2}\Delta\rho,\theta,\phi)(\rho+\frac{1}{2}\Delta\rho)^2\sin\theta\, d\theta d\phi$

Contribution of rear surface to the integral $= -f_\rho(\rho-\frac{1}{2}\Delta\rho,\theta,\phi)(\rho-\frac{1}{2}\Delta\rho)^2\sin\theta\, d\theta d\phi$

Add the above contributions, then normalize by $\Delta V = \rho^2\sin\theta\, d\rho d\theta d\phi$, and you'll find $\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 f_\rho)$.

Similarly, the contributions of the right and left facets yield

$$\frac{1}{\rho^2\sin\theta\, d\rho d\theta d\phi}\left\{f_\phi(\rho,\theta,\phi+\tfrac{\Delta\phi}{2}) - f_\phi(\rho,\theta,\phi-\tfrac{\Delta\phi}{2})\right\}\rho\, d\theta\, d\rho = \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi} f_\phi.$$

As for the top and bottom surface, we'll have:

$$\frac{1}{\rho^2\sin\theta\, d\rho d\theta d\phi}\left\{f_\theta(\rho,\theta+\tfrac{\Delta\theta}{2},\phi)\sin(\theta+\tfrac{\Delta\theta}{2}) - f_\theta(\rho,\theta-\tfrac{\Delta\theta}{2},\phi)\sin(\theta-\tfrac{\Delta\theta}{2})\right\}\rho\, d\rho\, d\phi = \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta).$$

Consequently: $$\vec{\nabla}\cdot\vec{f}(\rho,\theta,\phi) = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 f_\rho) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi} f_\phi.$$