Solution to Problem 17) Given that a unitary matrix A can always be diagonalized with the aid of a unitary matrix S, denoting the degenerate eigen-value of A by λ, we will have

$$A = S\Lambda S^{*T} = S\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} S^{*T} = \lambda SIS^{*T} = \lambda SS^{*T} = \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}. \tag{1}$$

Now, since the eigen-values of unitary matrices must be on the unit-circle in the complex-plane, the single eigen-value of A is given by $\lambda = \exp(\mathrm{i}\varphi)$. Consequently, the only 2×2 unitary matrices with degenerate eigen-values are in the form of $\exp(\mathrm{i}\varphi)\, I$.

An alternative approach to solving this problem is to start with an arbitrary 2×2 $ABCD$ matrix, then impose the unitary requirements on its four (generally complex) elements, as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

Equating the individual elements of the above matrices now yields

$$|A|^2 + |B|^2 = |C|^2 + |D|^2 = |A|^2 + |C|^2 = |B|^2 + |D|^2 = 1, \tag{3a}$$

$$AC^* + BD^* = AB^* + CD^* = 0. \tag{3b}$$

The first of the above equations requires that $|A| = |D|$ and $|B| = |C|$. The second equation, written in terms of the magnitudes and phase-angles of the four matrix elements demands that $|A||C| \exp[\mathrm{i}(\varphi_A - \varphi_C)] = -|B||D| \exp[\mathrm{i}(\varphi_B - \varphi_D)]$, which, in light of the first set of requirements, becomes $\varphi_A + \varphi_D = \varphi_B + \varphi_C \pm \pi$. The second half of Eq.(3b) imposes the same constraint on the phase-angles as the first half of the equation. All in all, the $ABCD$ matrix is found to be unitary if it satisfies the following constraints:

$$|A| = |D|, \quad |B| = |C|, \quad |A|^2 + |B|^2 = 1, \quad \varphi_A + \varphi_D = \varphi_B + \varphi_C \pm \pi. \tag{4}$$

Digression: It is *not* necessary to rely on both parts of Eq.(2) to arrive at Eq.(4). In fact, the first half of Eq.(3b) requires that $|A/B| \exp[\mathrm{i}(\varphi_A - \varphi_B)] = |D/C| \exp[\mathrm{i}(\varphi_C - \varphi_D \pm \pi)]$, while the first half of Eq.(3a) demands that $|B|^2(|A/B|^2 + 1) = |C|^2(1 + |D/C|^2) = 1$. Combination of these equations leads to Eq.(4).

We now write the characteristic equation of the $ABCD$ matrix in order to determine the conditions under which the eigen-values are degenerate. We will have

$$\begin{vmatrix} A - \lambda & B \\ C & D - \lambda \end{vmatrix} = \lambda^2 - (A + D)\lambda + (AD - BC) = 0. \tag{5}$$

The eigen-value degeneracy occurs when $(A + D)^2 - 4(AD - BC) = 0$, a condition that simplifies to $(A - D)^2 + 4BC = 0$. Substitution from Eq.(4) now yields

$$(A - D)^2 + 4BC = |A|^2 \exp[\mathrm{i}(\varphi_A + \varphi_D)]\{2\mathrm{i}\sin[(\varphi_A - \varphi_D)/2]\}^2 + 4|B|^2 \exp[\mathrm{i}(\varphi_B + \varphi_C)]$$
$$= 4\{|A|^2 \sin^2[(\varphi_A - \varphi_D)/2] + |B|^2\} \exp[\mathrm{i}(\varphi_B + \varphi_C)] = 0. \tag{6}$$

Equation (6) is satisfied when $|B| = 0$ and $\varphi_A = \varphi_D$. Together with Eq.(4), this result implies that the eigen-values of the unitary $ABCD$ matrix will be degenerate when $B = C = 0$ and $A = D = \exp(\mathrm{i}\varphi)$.