**Solution to Problem 14**) By definition, the square root of the matrix $A$ must satisfy the relation $\sqrt{A} \times \sqrt{A} = A$. Given that $A$ is diagonalizable as $A = \tilde{V} \Lambda \tilde{V}^{-1}$, we define the square root of $A$ as follows:

$$\sqrt{A} = \tilde{V} \Lambda^{½} \tilde{V}^{-1}.$$

Here $\Lambda^{½}$ is a diagonal matrix whose elements are the square roots of the eigen-values $\lambda_n$ of the matrix $A$. It follows that

$$\sqrt{A} \times \sqrt{A} = (\tilde{V} \Lambda^{½} \tilde{V}^{-1})(\tilde{V} \Lambda^{½} \tilde{V}^{-1}) = \tilde{V} \Lambda^{½} (\tilde{V}^{-1} \tilde{V}) \Lambda^{½} \tilde{V}^{-1}$$
$$= \tilde{V} \Lambda^{½} I \Lambda^{½} \tilde{V}^{-1} = \tilde{V} \Lambda^{½} \Lambda^{½} \tilde{V}^{-1} = \tilde{V} \Lambda \tilde{V}^{-1} = A.$$

In general, each eigen-value $\lambda_n = |\lambda_n| \exp(\mathrm{i}\varphi_n)$ is a complex number whose two square-roots are $\lambda_n^{½} = \pm\sqrt{|\lambda_n|}\, \exp(\mathrm{i}\varphi_n/2)$. Thus, there are two roots for each eigen-value of $A$, unless $\lambda_n = 0$, in which case there will be a single root only. Assuming that $A$ has $m$ non-zero eigen-values, the total number of matrices that can be identified as $\sqrt{A}$ is going to be $2^m$.

As examples, we use the following $2 \times 2$ and $3 \times 3$ matrices, both of which were diagonalized in Problem 8:

**Example1**:
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = ½ \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} -\mathrm{i} & 1 \\ \mathrm{i} & 1 \end{pmatrix}.$$

$$\sqrt{A} = ½ \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \pm e^{\mathrm{i}\pi/4} & 0 \\ 0 & \pm e^{-\mathrm{i}\pi/4} \end{pmatrix} \begin{pmatrix} -\mathrm{i} & 1 \\ \mathrm{i} & 1 \end{pmatrix}$$

$$\rightarrow \quad \sqrt{A} = \pm\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \sqrt{A} = \pm\frac{\mathrm{i}\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

**Example 2**:
$$B = \begin{pmatrix} 1 & 0 & \mathrm{i} \\ 0 & 2 & 0 \\ -\mathrm{i} & 0 & 1 \end{pmatrix} = ½ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ \mathrm{i} & -\mathrm{i} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\mathrm{i} \\ 1 & -2 & \mathrm{i} \\ 0 & 2 & 0 \end{pmatrix}.$$

$$\sqrt{B} = ½ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ \mathrm{i} & -\mathrm{i} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \pm\sqrt{2} & 0 \\ 0 & 0 & \pm\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & -\mathrm{i} \\ 1 & -2 & \mathrm{i} \\ 0 & 2 & 0 \end{pmatrix}.$$

$$\rightarrow \quad \sqrt{B} = \pm\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & \mathrm{i} \\ 0 & 2 & 0 \\ -\mathrm{i} & 0 & 1 \end{pmatrix}, \qquad \sqrt{B} = \pm\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -4 & \mathrm{i} \\ 0 & -2 & 0 \\ -\mathrm{i} & 4\mathrm{i} & 1 \end{pmatrix}.$$

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